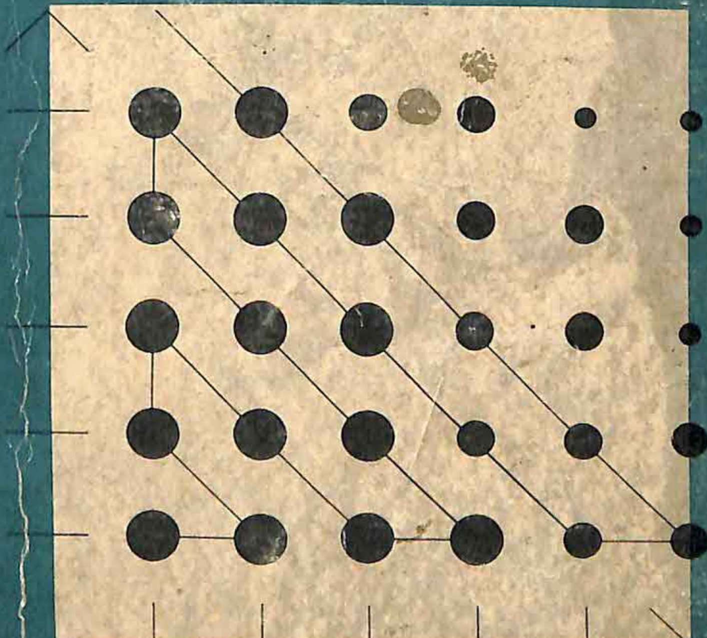
New trends in mathematics teaching

Volume IV

The teaching of basic sciences

Mathematics



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Volume IV

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Preface

The worldwide exchange of ideas and information plays an important role within Unesco's programme to promote innovations for the improvement of mathematics education. One of the Organization's contributions to this international exchange is the publication of *New Trends in Mathematics Teaching*, within a broader series, *The Teaching of Basic Sciences*.

This volume, the fourth on the subject of mathematics education, is to a large extent based upon the preparation for and proceedings of the Third International Congress on Mathematical Education (ICME) held in Karlsruhe, Federal Republic of Germany, in August 1976. This Congress, the largest International meeting of its kind, brought together nearly 2,000 mathematics educators from seventy-six countries. Efforts to broaden participation from developing countries resulted in the attendance of educators from eighteen countries which had not been represented at previous Congresses.

The introduction which follows explains the planning of the thirteen chapters of this volume and the steps taken to ensure that the contents reflect worldwide trends in mathematics teaching. The international community of mathematics educators, as represented at Karlsruhe, was in general agreement with these papers, which show the common trends and processes of mathematics teaching. The present volume is, however, not a statement of what should be taught; this will depend upon each country's social, economic and cultural goals and resources.

All those who are actively interested in the improvement of mathematics education should find this volume useful; practising teachers and teacher educators, teachers-in-training, members of examination boards and teachers' associations, school inspectors, curriculum development and research workers, and especially those responsible for making decisions in respect of mathematics curricula.

Selected chapters could also be used as resource material in mathematics education workshops and conferences. Problems of a national educational system or those common to a group of countries can be studied in the light of the international body of ideas and information contained herein.

Unesco wishes to express its appreciation to all the many persons who helped to prepare this volume – authors of papers, all of whom agreed to remain within the boundaries set by the editorial board; panel members for constructive criticisms of the draft papers; Congress participants who contributed to the discussions; the International Programme Committee and the Federal Republic of Germany Organizing Committee of ICME. Special thanks are due to the editors of this volume (Professor B. Christiansen and Professor H.G. Steiner) and their co-editors (Dr. T.J. Fletcher, Professor A. Revuz and Professor L.A. Santaló) for their competence, care and patience in editing the papers of thirteen authors.

The views expressed by the various contributors are the responsibility of the editors and authors and do not necessarily reflect the views of Unesco. The International Commission on Mathematical Instruction was entrusted with full editorial responsibility for this volume.

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Introduction

For several years past, the United Nations Educational, Scientific and Cultural Organization (Unesco) and the International Commission on Mathematical Instruction (ICMI) have co-operated on a number of important tasks. Among these co-operative activities, the development of the series *New Trends in Mathematics Teaching* has been of special prominence. Volumes I and II, prepared respectively in 1966 and 1970, contained various articles on key topics in mathematics education, written by internationally recognized specialists. These two volumes were well received, since they brought extensive information about innovative efforts, but it may have been difficult for the reader to distinguish individual contributions (which considered general problems to be coped with) from major trends. Unesco therefore decided that future volumes should be composed of scholarly analyses of such problems and trends, so that the contents of the volumes would correspond more closely to their titles.

Accordingly, volume III followed a new pattern, based on the conclusions of a two-week meeting at which mathematics education specialists discussed those chapters which were still only in tentative form. It was felt that additional action should be taken to ensure that trends in the development of mathematics education were described on a sound and appropriate basis.

Unesco therefore arranged to discuss thoroughly the possible composition of volume IV. This discussion took place in Paris at the end of 1974 at an Ad Hoc Advisory Meeting convened by ICMI on behalf of Unesco. Among the participants were staff members of Unesco and members of the Executive Committee of ICMI. The structure of the previous volumes was carefully analysed, and new plans were made to identify both major problems in the field of mathematics education and the direction and intensity of changes taking place in that field. As expected, Unesco and ICMI were in full agreement about the purposes of the volume, and the meeting further clarified the principles for its development and the strategies and working methods by which the necessary co-operative support from mathematics educators around the world could be obtained and facilitated.

Unesco requested that ICMI assume full professional responsibility for the development of the volume in accordance with the general plan identified at the Paris meeting. It was intended that work on the publication should be linked to the professional preparations for the Third International Congress on Mathematical Education to be held in Karlsruhe, Federal Republic of

Germany, 16-21 August 1976. It was agreed that meetings of authors with editors and other experts would be necessary both before and after the Congress. As a first step in these co-operative efforts, an editorial board was established to assist ICMI in fulfilling the Commission's obligations towards Unesco. The two Vice-Presidents of ICMI became members of this board and chief editors of the volume. Three other mathematics educators served as members of the board, with special responsibilities related to the English, French and Spanish language versions of the volume.

The International Commission on Mathematical Instruction welcomes this occasion to describe how the envisaged procedures were actually carried out. It should be emphasized that the planning meeting in Paris at Unesco Headquarters was of great importance for the subsequent work, and that these plans greatly benefited from the co-operative spirit of the National Organizing Committee for the Karlsruhe Congress.

Three aspects were of major importance in the development of the volume: first, the scope and purposes of the work seen in relation to the general pattern of its execution; second, the details of the system within which each chapter was established; third, the specific difficulties and constraints inherent in the task itself, namely establishment of survey-trend reports. Knowledge of these three aspects will help the reader to see the ways in which this volume may be useful as an instrument in the hands of the mathematics educator or teacher.

The scope, purposes and development of this volume

It was decided during the Paris meeting that this volume should deal with thirteen broad themes of general educational interest for mathematics teachers and educators. Six of these themes divided the field of mathematics education according to levels, ranging from pre-school to adult education, whereas the remaining seven made a "vertical" division of the field in question, so that each theme would be of interest for the work at all levels, e.g. curriculum development, evaluation, research. The final selection of the themes appears as the chapter headings in the list of contents.

For each of these themes a report was developed by a recognized specialist in the subject. The report was next presented to a section – dealing with the same theme – at the Third International Congress on Mathematical Education (the Third ICME). Finally, the report was developed further into a chapter of this volume, and in this development due consideration was given to all relevant activities of the Congress section concerned.

The contributions of each specialist, first as reporter and later as author, went through a number of stages which gave the reporter/author an opportunity to receive feedback from other specialists in his theme. The purposes governing this whole working process were as follows: (1) to give, as nearly as possible, an objective presentation of the developments and trends, as revealed by the literature and other documents; (2) in the light of these findings, to identify and analyse the relevant problems; (3) to suggest actions which might meet challenges or solve problems; (4) to present an appropriate bibliography for the theme. In general, all authors, in their dealing with these four tasks, expressed personal views in different degrees. Thus, in (1) above, a high degree of "objectivity" was possible, although conditions varied from chapter to chapter. In (2), the subjective judgement of the author clearly played an important role. In (3) and (4), but especially in (3), the choices of the specialist depended to a great extent on his experience, judgement and views.

The above procedure and purposes indicate that the development of *New Trends in Mathematics Teaching*, volume IV, was indeed an ambitious and difficult task, which could only be accomplished in a tentative and approximate sense. At the Ad Hoc Advisory Meeting at Unesco it was seen that the development of an appropriate volume would depend on an expansion of the existing network of international co-operation. This co-operation would be a means to counteract two major risks which are implied by the intended scope and purposes of the volume. One risk was the danger that each of the thirteen themes might be considered in isolation from the other themes, which could be counteracted by bringing authors together at several stages of the working process. The other problem was the danger that the developments and tendencies of a particular theme might be judged in relation to one specific context only. The provision of interaction between the reporter/authors and specialists from a number of countries around the world would counteract this tendency.

Some details concerning the development of the chapters

The first step to be taken by ICMI was the identification of the team of thirteen reporter/authors. The second was the establishment, for each theme, of an International Advisory Group of specialists on that theme (10–15 members), broadly representative both with respect to countries or regions and to the various aspects of the theme. The members of this group had several crucial functions: first, to provide the reporter/author with information on other national or professional contexts; second, to give constructive criticism at several stages of the working process; third, for those members of the Advisory Group who were able to attend the Congress, to join the core group of a Panel to discuss the report at the Congress section on their theme.

The International Programme Committee of the Third ICME became the instrument under which these early plans developed into a practicable form. This process involved the co-ordination of the development of the chapters with the preparation of the reports to be presented at the Congress. The Committee comprised some twenty specialists from fifteen countries, and counted representatives of the editorial board of this volume as well as the National Organizing Committee for the Third ICME. For the theme of each Congress section, available international expertise was screened by a sub-committee of the International Programme Committee, and personal contacts were established accordingly through correspondence and direct consultations. As a result, the team of reporter/authors was finally established in June 1975, and the thirteen International Advisory Groups were forming by the end of 1975. By the middle of 1975, "New Trends IV" began to take shape. Some of the major stages in the development of the volume are indicated below.

First, the chief editors of the volume, acting on behalf of ICMI and Unesco, drew up a list of sub-themes and aspects which should presumably be part of the reports and of the corresponding chapters. These brief statements concerning the thirteen themes were sent to all reporter/authors together with basic information on the intended volume.

Second, each reporter/author developed an open-ended outline of his intended report. The purpose was *not*, at this stage, to develop a summary of the report, but to make views and intentions open to comments and criticism, and to indicate working principles and criteria for identification of trends or suggested actions.

Third, these outlines were sent to all authors and to the editors, to the International Programme Committee, and to selected German mathematics educators who would each serve as a *co-ordinator* of the contributions at one of the Congress sections. This group of about 30 specialists was then brought together for a three-day meeting in December 1975 at the Mathematics Research Institute in Oberwolfach, which was made possible due to financial support from the Volkswagen Foundation. The outlines were here subjected to a thorough discussion, which revealed the many cases where overlaps would be necessary and even fruitful.

Constructive criticism during the meeting resulted in a number of changes and adjustments at this early stage of the development of the reports. Thus, the discussions in Oberwolfach had a great influence on the subsequent work.

Fourth, the reporter/authors, acting on recommendations made during these discussions, drew up new outlines of their reports. These outlines, representing a closer approximation of the intended result, were sent to all reporter/authors, to the editors and to the members of the International Advisory Groups, with a request that feedback be given to the individual reporter/author. The following period – the spring of 1976 – became a very fruitful time in which professional advice, comprising information and criticism, furnished the reporter/authors with background for the next important stage in the working process.

This fifth and major stage was the development of the *survey-reports* (in preliminary form) and their abstracts. The reports were sent to the same group of persons mentioned above, whereas the abstract was given (in an English, French or Spanish version, as appropriate) to the participants in the respective Congress sections. Accordingly Panel members were able to base their interventions on the complete (preliminary) report. Furthermore, authors and editors had an opportunity to become acquainted with the total volume in its preliminary form. Preparations were then made for the sixth crucial step, consisting of several inter-related activities at the Third International Congress.

In August 1976, the themes of the chapters were studied and discussed as part of the activities of the thirteen sections at the Karlsruhe Congress. The number of participants in the sections ranged from about one hundred to some three hundred, since most of the1800 participants attended the meetings of two of the thirteen sections. The programme of each of these (comprising two half-day sessions) was as follows: presentation of the *survey-report* (based on the *survey-abstract*); selected *short communications* related to the theme; *discussions* of the theme, including contributions from Panel members and participants in the section. The reader will find a summary of the activities of each section (as well as a list of all short communications) in the *Proceedings of the Third International Congress on Mathematical Education**).

The activities in Karlsruhe brought the reporter/author into personal contact with his Panel, which was in most cases nearly the same as his International Advisory Group. At a concluding meeting, after the last session, the Panel members and the co-ordinator gave their final comments about the preliminary report specifically from three points of view: a) whether the factual picture presented was an accurate and balanced statement of affairs; b) whether the nature and urgency of the problems and changes were properly identified; and c) whether any action programme recommended seemed to be sound and practical.

Finally, after the Congress, the authors met with the editorial board for a two-day meeting on the development of the survey reports into chapters of this volume. The meeting considered, theme by theme, the state of affairs in the light of all the information and feedback which had become available during the foregoing long process. It was emphasized that, although the individual author in the final work on his chapter was to keep this total input clearly in mind, each author would have personal and professional responsibility for the development of his final paper.

As was perhaps to be expected, the finalization of the chapters proved in several cases to be a very time-consuming task. Many details had to be checked in the light of new information obtained, and co-ordinating adjustments had to be made by means of further co-operation between authors, Panel members and editors. The resolution of such difficulties, which are further elucidated below, required considerable time and therefore delayed the publication of this volume.

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Difficulties and constraints of the task

Is it possible to identify "new trends" in mathematics teaching? The difficulties inherent in the task become even greater when the purpose of the volume, as required by Unesco, is to "present an accurate, comprehensive and balanced picture of mathematics education around the world – what is going on, what difficulties are being experienced and what needs to be done – but written basically for anyone who is interested in what is going on in mathematics education around the world, particularly those interested in and/or able to promote innovations in this field."

Certainly, when the process of mathematics education in a selected country is considered over a certain period, changes are seen in a number of respects, and the direction and intensity of such changes, *trends in the developments*, are also seen. However, the *object for study* – mathematics education – is a complex and multi-faceted field, and its themes and sub-themes are closely inter-related. Moreover, the process of education is interwoven with the development of society, and hence is dependent on socio-cultural and socio-political aspects. Not only does the subject's perception of the object for study depend on his personal views and values, but inherent in the object itself are value-systems and value-judgements. Accordingly, even if the reviewer of the developments consciously attempts to be impartial and objective – which he certainly must – his personal values and views will influence his choice of aspects and factors to be analysed, as well as his insight into such selected items and his judgement of them.

Such difficulties were often encountered in the development of this volume. For example, when the reporter/author needed to consider a problem in different socio-cultural or national contexts, he had to base his comparison on a few arbitrarily-distributed views expressed by other specialists, who in turn were influenced by their specific contexts, and who may have described the problem in their own or in other countries. Awareness of these difficulties is, however, the first step to bringing them under some control. Moreover, the consideration of changes in mathematics teaching "around the world" necessitates that the identification of trends be limited in scope to such cases where it is universally accepted that a common tendency exists in the developments. Another way to confront these difficulties is to consider trends in specified contexts (e.g. national, institutional) for which broad agreement has been reached.

The problem of being "objective" and the dangers of identifying trends with limited background material – or in complex domains to which the concept of tendencies is not applicable – were carefully considered by the authors at the meetings in Oberwolfach and Karlsruhe. It was clearly recognized that the consideration of trends would have different interest and value in the different chapters. It was agreed that, in all cases, important parts of each chapter would consist of statements of the author's own view of the process of change, and that it should be made clear to the reader when personal views and proposals were expressed.

It is hoped that the danger of undue isolation of topics, which is implied by the division of the theme of this volume into the thirteen closely inter-related sub-themes dealt with in the chapters, has been counteracted by the following measures. First, each author – as far as conditions allowed – developed his chapter as a self-contained unit. This arrangement should aid the reader who is interested in only selected chapters, and the reader of the complete volume should benefit from the treatment of the same sub-theme from different perspectives as regards priority and detail according to the context of the chapter in which it appears. Next, the extensive co-operation between authors at the authors' meetings and by correspondence helped to avoid undue separation of themes. Third, in the case of the division into levels (corresponding to the first five chapters), the age-levels to be considered in each chapter were deliberately chosen so as to overlap.

Authors were requested to write at a level of interest for mathematics teachers and educators, but not to simplify matters unduly nor to write popular accounts.

Acknowledgements

The International Commission on Mathematical Instruction greatly appreciates the collaboration of long standing between the Commission and the United Nations Educational, Scientific and Cultural Organization. The Organization and the Commission have a number of common purposes in respect of their activities in mathematics education, and the co-operation involved in preparing this volume is but one example of how the professional expertise of ICMI and Unesco are combined for the attainment of common goals. Accordingly, the following acknowledgements, expressed on behalf of ICMI, are also expressions of appreciation from Unesco.

ICMI greatly appreciates the generous professional assistance which has been offered by numerous colleagues around the world who have shared in the preparation of this volume. In this group, it is appropriate to single out certain specialists who have made particular contributions: the members of the International Programme Committee for the Third ICME, the members of the National Organizing Committee for the Congress, and the members of the International Advisory Groups and Panels for the themes of the chapters of the volume. The last-mentioned colleagues, who agreed to interact and co-operate with the reporter/authors in the preparation of the publication, either by correspondence or at the section meetings in Karlsruhe, are listed at the end of the chapter to which their work contributed so fruitfully. Special thanks are also due to our colleagues on the editorial board, T.J. Fletcher, A. Revuz and L.A. Santaló, whose continuous efforts throughout the working process have been indispensable to ICMI.

But first and last, the gratitude of ICMI goes to the thirteen colleagues who took upon themselves difficult and time-consuming tasks as reporters and later as authors, and who generously offered their time and their professional expertise throughout the long-term working process described above.

In addition, ICMI expresses its appreciation of the important indirect assistance from universities and departments around the world who have shared in the work through generous secretarial assistance to the professionals working for this cause. It is appropriate to mention in this connection the special support obtained from the Institut für Didaktik der Mathematik of the Universität Bielefeld and from the Royal Danish School of Educational Studies.

On behalf of the International Commission on Mathematical Instruction and its President, Professor S. Iyanaga,

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Chapter I

Mathematics education at pre-elementary and primary levels

F. Colmez

INTRODUCTION

In this chapter we shall try to define the chief tendencies in the mathematical education of children from the beginning of their school-life up to the age of 10-12 years. In many countries, this age no longer indicates the end of compulsory schooling, but it is still very often the time when a change occurs in the pattern of education; this is the age, according to Piaget, which marks the end of the stage of concrete operations.

At all events, the years of elementary education are still the years, within the context of school, when young children first come into contact with mathematical activity, and on that account is of paramount importance for their future, and especially for the attitude they will adopt towards learning mathematics. Moreover it is at this point that the various problems raised by the teaching of mathematics are more easily formulated and observed.

Current development is motivated by two desires which are complementary but different in character: (1) *to enhance the content of education* by promoting the standardizing and simplifying power of mathematical thought, with the object of improving each individual's level of understanding and grasp of an environment full of mathematical situations; (2) *to improve the learning process* of every child and to introduce the study of mathematical ideas at the most appropriate moment.

During the sixties, it was essentially the first idea which motivated the will for reform at all levels of education. It was a question of bridging the gap between mathematics taught in school and mathematics as developed by mathematicians. An attempt was therefore made to pass on the general unifying ideas at more and more elementary levels of teaching.

During the seventies, the move from a stage of isolated experiments to a stage in which experiments had already been undertaken, or where planned modifications were about to be implemented, led to a situation in which more importance was given to the second idea.

It is the results of this change in priorities which form the background of the new trends which we shall attempt to analyse in this chapter.

1 GOALS

1.1 Mathematics or arithmetic

In many countries the word "mathematics" has been introduced at the elementary level of education. The prestige which this word carries and the dread it inspires have very often led a misinformed public to think, first, that the arithmetic taught previously was in conflict with the mathematics taught now and that, secondly, the latter would be taught at the expense, and even in place, of arithmetic. This idea is false; and to avoid this mistake some countries have preferred to use the terms "pre-mathematics" or "mathematical arithmetic".

This dispute over words reflects in fact the present varying trends in elementary education, from the point of view of goals as well as of content and methods.

Although the range of goals set out for elementary teaching in mathematics varies widely from one place to another, the world-wide trend is to extend these goals. In many countries the citizen's basic training in mathematics is no longer ensured by the elementary school alone, and in any case the demands of working and professional life develop along much more complex lines. The result is that the fundamental goal of the elementary school is no longer to help children acquire techniques for solving well defined problems within a restricted area, but to try to provide them with a correct approach and a real understanding of the mathematical concepts associated with these techniques, and a solid base for their continuing education.

1.2 The value of attainments

In many countries, the role of selection and orientation in elementary education is now less important; even officially this role has often disappeared, but the facts do not always show this to be the case; it probably still exists in a more insidious way. This is closely linked with the principal fact which distinguishes the present time in many industrialized countries; owing to its rapid development, the educational goals of the elementary school are no longer clearly discerned by its most interested parties: parents, pupils and teachers. These goals can no longer be confined simply to those of acquiring knowledge and skills within a limited area (essentially the four operations), and the problem thus raised, of assessing the proficiency of the pupils, has probably a great deal to do with the confusion which is often noted between aims and measures taken. Thus it comes about, for example, that the inadequacy of information given to the public and of training provided for teachers, has led, in some areas, to the following misconception in the interpretation of the new content-matter: a lot of time is devoted to the study of sets and relationships, and this study is considered as a goal in itself, and not a means of attaining, among other things, a better understanding of numbers. There is less time left for numeracy and, as teaching in this area has not greatly improved, the effects are felt in the pupils' grasp of computation. These sorts of mistakes are gradually being corrected, but they have given rise to controversy between the advocates of a return to former goals and "methods which have proved themselves", and those people who accepted this decrease in importance attributed to activities involving numerical computation and justified it by maintaining that since elementary teaching involves less streaming by ability, it should concentrate less on the principal criterion used in streaming, which is the evaluation of computational skills.

These two positions are equally unprofitable and the solution consists rather in entirely reconsidering the problem of acquiring skills, by transferring it to a much wider context of the learning process, which includes simultaneously, competences, understanding, knowledge and skills.

This is the emerging trend at present: not only to maintain a good level of attainment in arithmetical proficiency and skills, but to extend this aim to other fields and thus create in the pupils fresh abilities, attempting especially to leave the way open for later adaptations.

1.3 Development of an exploratory attitude

The preceding sections relate to the subject which is taught; the aims we shall now attempt to define relate to the child who does mathematics. In this field, the most interesting trends are based on the idea that there is no basic difference between the way in which a child acquires knowledge and the way in which a mathematician creates it and that, as a result, the teaching of mathematics should be considered mainly as a rediscovery. In point of fact it may be noted that the following trends are being adopted only very slowly.

We put in first place a goal which, although not entirely new (it has been found as a regular item for a long time now in official texts), is now considerably strengthened by the fact that efforts are being made to find effective means of attaining it: the development of the attitude of research in pupils.

Thus pupils are presented with situations in which there are several possible approaches to the solution, at different levels of ability or formal techniques, while care is taken not to discourage the various endeavours nor to favour one method of solution more than the others.

Each pupil is capable of doing something and can eventually move from one level of solution to another if he is stimulated by discussion within the class.

This goal assumes a considerable change in the conception of the nature of teaching. Previously, the learning of techniques for solving closely defined problems whose terms include the few key words giving the pupil the clue as to which mechanical process to apply often reduced the pupil's role to that of a computer which has to learn programs. This conception of teaching has unfortunately not yet disappeared; it often produces an emotional blocking in the pupil who is afraid of not knowing the answer, and it leaves him helpless when he finds himself confronted with a problem which is not closely defined. On the contrary, the development of an exploratory attitude is a goal which has both a social and a psychological basis. Socially, the variety of questions in which mathematics occurs makes it impossible to provide a child with a sufficiently wide range of techniques to allow him to confront all the problems he will meet in the course of his life (most of which are unknown at the present day); and psychologically, the pupil does not learn mathematics by contemplating the complete mathematical structure, but by a reasoned dialogue, which it is the teacher's duty to foster. This goal can be stated more clearly and succinctly as follows:

- to foster the natural curiosity of children and their desire to understand by presenting them with situations where the procedure to apply depends on a mathematical model within their grasp;
- to leave children to develop their own strategies of research;
- to allow each child to succeed in order to encourage his research in the future;
- to encourage pupils to appeal to their previous skills and abilities in order to explore new situations;
- to stimulate children to suggest new topics of activity of their own accord.

1.4 Intellectualization of elementary teaching

Complementary to the goal in 1.3 there is a trend, less well-known and less clearly expressed in official texts, towards an intellectual approach in elementary teaching. By this we mean here: the introduction of more precisely defined mathematical concepts; a more global approach; and a better structured organization of knowledge. It is by means of reasoned steps taken between the different levels of processes and thought that this build-up is achieved, by progressively introducing new ideas. For example, in a piece of work on the measurement of areas there will be different types of manipulations spread over a period of time which will facilitate, in succession, the choice of criterion against which to compare objects, the definition of the concept of unit of measurement, the construction of instruments of measurement, the improvement of these instruments, the production of objects of a given measurement, the definition of the concept of area, the inter-relationship of the different sorts of area, the construction of certain formulae giving the areas and the volumes of simple geometrical objects.

Also under the heading of intellectual approach comes the research into regularities, similarities of different situations, common models, that is, the discovery and implementing of the unity and the economy of thought which make mathematics so useful. This is a very ambitious goal and the means for fully achieving it are being developed; much research is being carried on in this area.

However, it seems that children are capable at a very early age of creating, handling correctly, and modifying according to their needs, the simple mathematical language necessary for describing and studying well-chosen situations, while respecting the principal criteria which constitute the efficiency of this tool of investigation and communication: intelligibility, brevity, accuracy, relevance, normality.

In particular, the deliberate use of symbols to indicate objects or constants, unknowns and variables, encourages them to construct an effective instrument for solving problems. In the same way, children are able to grasp the dynamic aspect of mathematics (discovery, construction, reorganization), and one can perceive in their problem-solving strategies moments of intuitive as well as analytical thought.

This goal is indeed one of the most important for the future. Like the goal in 1.3 (to develop an exploratory attitude) it is inspired by the deep concern to respect the intelligence and personality of children. Moreover, it is fundamental if an attempt is to be made to reduce the current split of industrial societies into two groups of people: one group having a few scraps of technical competence with little logical organization, the other group having a well-structured knowledge.

1.5 Mathematics as a collective creation

Increasing attention is being paid to the aspect of group creation by the class at the elementary level of mathematical teaching. Mathematics is not the sole activity in which learning to work together and to communicate with each other is an important part; but in mathematics, especially in the early stages of teaching, all the ideas and concepts can be constructed by the children from activities devised for them.

Such a conception of teaching also aims at giving the child opportunity to appreciate both the futility of knowledge which is not shared, and the social need to ascertain that one's partner in a conversation possesses the same model as oneself if communication with him is desirable, and as a result, to understand the benefit there is in helping him build that model.

This aim is very difficult to achieve; it requires varied teaching methods, adapted to each pedagogic situation or intention. It demands that the teacher become much less apparent as a giver of information and much more available and alert as an organizer and animator; this requires much greater professional ability. Finally, the size of classes must be adapted accordingly (neither too many nor too few); the optimum size is probably somewhere between 20 and 25 pupils.

For all these reasons, this goal may seem a Utopia in the current teaching situation; this is yet another argument for increasing research into the exact conditions necessary for its achievement.

1.6 Pre-elementary teaching

The idea of introducing activities with mathematical content into the nursery school has been apparent for some years.

First of all it must be noted that the status of pre-elementary education varies greatly from place to place. In many cases this stage of education does not even exist; if it does exist, the attainment of skills is not an aim of prime importance. Those who teach at this level receive a training which often differs from that of primary school teachers, having little mathematics; they concentrate much more on children's affective problems rather than on the cognitive problems. For all of these reasons, no far-reaching change can be introduced rapidly, and even research is more difficult in this field.

However, reasons abound for research at the level of pre-elementary education. Psychologists have shown that the intellectual maturity of a child depends largely on the activities in which he has the opportunity to participate. We cannot exclude the hypothesis that the absence or delay of certain activities creates an insurmountable handicap for the child.

One goal of pre-school teaching should therefore be to devise activities for children which they do not necessarily find in their family environment, and to exploit them by helping the child to recognize the relevant objects of thought and to construct his own patterns and structures. Beyond the activities of sorting and classifying which are fairly generally practised, many children's activities relate to mathematics in the areas of time, movement, speed, spatial models, measurement, geometric-type creations, relationships of all kinds, symbolic notation, ideas of quantity, etc. Some of these areas are consciously dealt with in pre-school education in some countries; however, in most countries no clear trend towards an intellectual approach in pre-elementary teaching can yet be discerned, even if the social importance of the latter is increasing.

2 CONTENT

The public has often thought that the content of elementary education in mathematics has changed completely. In actual fact, the modifications are on the whole modest, and arithmetic remains the central subject of mathematics teaching at elementary school level.

2.1 Operational techniques

The implementation of the reform in mathematics teaching at the elementary level has given rise to research on the problems linked with the attainment of operational techniques, and has afforded an opportunity of once more laying open to doubt a large number of generally accepted ideas.

The majority of people and many teachers still think that for each arithmetical operation there is only one possible operational technique – the one with which they are familiar. Moreover, the operational technique is usually clearly separated from the meaning of the operation. This expression is taken to mean, beyond the mathematical definition, the recognition of situations where a certain operation leads to the solution of a problem.

This gives rise to a division in activities: on the one hand, there is the study of meaning, on the other, the study of technique. In turn, the study of technique consists of two parts: one is the repertoire to be committed to memory (the operational tables), the other the computation program which, broken down into a sequence of sub-programs, is taught progressively, beginning with particular cases in which the algorithms are simplified (e.g. simple additions or multiplication by a one digit number). In fact, it is noted that each operational technique is the result of a balance between the algorithm which is used and the repertoire which is known. If the repertoire is limited, the algorithm required is long (adding on the fingers amounts to using the one times table only). Conversely, a very full repertoire allows the algorithm to be shortened.

Traditionally, in every country, the simultaneous choice of an algorithm and a repertoire is the historic result of an overall optimization of the effort of memorizing and the time spent on each calculation.

To help remove the mystique of the subject among adults, it would be interesting to have available a collection of the different technqiues used in various countries. This collection could also be of use in the training of teachers.

A similar role can be fulfilled by computation in number systems to different bases. But the idea of improving children's learning of algorithms by varying the number base has proved very disappointing in practice; one of the reasons explaining this is the fact that for each base the child must, if he is to use the necessary repertoire, either memorize it in this base, or transfer it from another base, or consult it having written it down; as, on the other hand, the construction of the algorithm shows little change, the total task of the pupil becomes all the more complicated.

Current research is attempting to encourage integrated, progressive learning, using meaning, repertoire, and algorithms at one and the same time. The range of an operation's meaning, limited at first, extends progressively by building in certain results, that is by establishing at first an algorithm using part of the range of meaning as a repertoire. The simultaneous development of the algorithm and the repertoire, brought about by new situations, allows an extension and strengthening of the meaning of the operation. The development of the repertoire is produced both by memorizing new results, judged more appropriate, and discarding others little used; at the same time, the structuring of this repertoire takes place in accordance with the algorithm, and in return leads to simplifications of it. At each stage a balance is established, and the development is guided by considerations of economy. The algorithms used are not necessarily the same in a mental computation as in a written one; mental computation plays an important role in reinforcing what has been memorized and in simplifying algorithms.

To sum up, it can be said that far from discarding the study of computation, elementary teaching seeks to promote a better understanding of the operations and a better control of the operational techniques by new methods which encourage a reduction in the time devoted to this study and avoid tedious repetitions.

2.2 Natural numbers

The introduction of sets in the elementary school is one of the things which has stirred up most criticism; some of this was and still is well-founded, as we show in the following analysis.

The word "set" had been introduced in a naive sense by certain promoters of the reform, in order to be able to speak more easily and more consistently of collections of objects whose comparison leads to the concept of number. But, on the one hand, this word made some good people shudder, since they appeared to take the word in the strong meaning of *set theory*; on the other hand, teachers wished to express ideas in concrete terms by representing sets by stereotyped drawings called Venn diagrams in many text-books, thus creating automatic responses in the children.

In the same stereotyped procedure, the use of one to one correspondence to compare two sets is done with the aid of lines linking the elements of sets inside two potato-like figures, which is useless on small sets as the comparison is done by the eye or by mental calculation (so that the drawing becomes an end in itself and is no longer a means), and which on sets which are too big produces an inextricable drawing; in the same stereotyped practice, the sum of two numbers can only be considered by the drawing of three "potatoes" of which two are inside the third etc. . .

This procedure does not allow any distinction between different levels of abstraction: collections of objects, representations of sets, classes of sets, numbers, and the writing of numbers. Children, not having touched on the process of abstraction, are lost at the level of writing down the numbers, which seem meaningless to them. To remedy this difficulty one comes back down the scale of abstraction by expressing written work in concrete terms with the help of such objects as multibase blocks.

One may note a tendency to correct this mistake by constructing the natural numbers in stages, each stage requiring the introduction of methods at different levels of abstraction; for example, comparison by one to one correspondence and simple enumeration of numbers up to 15-20, comparison of "bundles" and written recordings of numbers up to 40-50, systematic "bundling" and numeration beyond 60. Each new tool, introduced because of an extension, can also be applied in previously explored fields. Thus a coherent construction is reached, where all aspects of number are taken into account and where confusion between levels is avoided.

2.3 Computation and the extension of the concept of number

A trend has been seen developing in the teaching of computation which may be termed *structuralist*, which emphasizes mathematical structures with a view to changing the teaching of arithmetic with the help of general ideas of transformations, relationships, etc . . . Now it no longer seems that the activities thus contrived encourage pupils to reach the fixed goal in the expected way. This does not mean that such activities are uninteresting – on the contrary – but their interest lies in another direction.

For example, hoping to help in the construction of (Z,+), the study of several finite groups is introduced (mainly with 4 or 6 elements) in which the elements are one-one mappings of a set onto itself, and where the law of combination is composition of mappings. The dynamic character of the mappings is brought out by arrow diagrams or with the ideas of "function machines".

In the same way, operators for addition and subtraction are introduced as functions in N. But for the composition of operators analogy no longer works: for example, the composition of the operators "subtract 3" and "add 2" does not give the function "subtract 1". (The domain of definition is not the same.) If the computations are clear at the level of counting, their transfer to the function level is ambiguous.

This difficulty can only be surmounted by establishing written conventions which are not linked with precise concepts while the children continue to do their computation at the level of numbers. In classes where this procedure has been carried out very successfully, it is in fact because the children have extended the domain of numbers into negatives, by depending on other situations such as displacements on graphs; but the clarifications produced by operators do not correspond to the process implicitly followed.

These difficulties lead to a regression in the teaching of algebraic structures as such, but favour the study of certain algebraic properties which make up their components, like number relationships (especially congruences), the properties of the operations (for example, the distributivity of multiplication over addition), functions and in particular the affine functions which come into numerous problems and whose various representations by tables, graphs, etc. stimulate children to carry out a great deal of research.

From a similar viewpoint many activities devised for children may be classed under the heading *combinatorics*. This allows the teacher to present open situations adapted to the pupils' capabilities for investigation and it is up to them to organize their work, to discover through systematic exploration some interesting properties of numbers, and in many cases to do actual demonstrations. Although some teachers restrict these possibilities to systematic tree-constructions, thus creating automatism in their pupils, generally the proficiency acquired by the pupils during exercises of a combinatorial character encourages them to think out their computations in a better way.

The main content of elementary teaching in all countries is still the introduction of natural numbers and the study of some of their properties.

Many countries introduce negative numbers at a fairly early stage in elementary teaching; others do not do so at all.

It would appear that in some countries the study of ratios, (usually positive ratios) is less intensive than before, as more energy is devoted to those particular ratios known as decimals (these encourage a better approach to $\mathcal{R}$ and are much more important in the sciences and contemporary life). It is likely that the widespread use of pocket calculators will promote this trend.

2.4 Geometry

Great changes have come about in the teaching of geometry. Some years ago this teaching was usually normative (essentially a few descriptive definitions of figures), and formal (many computational problems of measurement and conversion). A new kind of teaching is taking shape at present, incorporating the skills of the former. Two trends stand out.

One is a structural trend in which a simplified model of plane affine geometry is studied, starting with notions of parallel lines and using parallel linkages and squared paper. The relationships of incidence of parallel lines are studied as well as certain simple transformations (translations, other displacements and similarities). The labelling of points by pairs of numbers then helps to enrich this model by developing analytic geometry on D^2 with an approach to $\mathcal{R}^2$. A distance is introduced which is sometimes called "taxidistance" which helps to set problems of the type: find the shortest distance between one point and a set of points, find the median point, etc. The solution of these problems requires numerical investigation. Many other activities are possible along the same lines.

The second is an exploratory trend based on active participation. Children are given objects or plane figures to classify and the study of classification criteria helps to define certain concepts such as polyhedrons, polygons, convexity, etc. Problems of reproducing the object introduce some properties of symmetry, of the incidence of parallel lines, and of some plane transformations, which also come into constructions of tilings of the plane, and regular polygons or polyhedrons. The use of squared paper helps to simplify certain constructions, facilitates their solution by the introduction of numbers, and stimulates arithmetical research. The use of assemblies of cubes affords an opportunity of setting and solving problems from the point of view of shadow cast and representation with the help of projections, etc. Many other activities may be introduced, stimulated by the environment.

The structural trend being the earlier one, it appears that the second trend is developing in some countries as a reaction against the first, supported by the argument that the structural trend portrayed a conception of geometry which was too algebraic and did not sufficiently emphasize the concepts most useful to the child for his spatial awareness. In fact, these two trends do not seem incompatible (the activities based on geoboards, for example, hinge the two together).

As for measurement, a fairly wide-spread trend is to devise activities constructed around the idea that for each particular magnitude a measuring instrument allows a number to correspond to each object (with or without approximation). Studies are being made of what happens when the measuring instrument is changed, and what sort of information is given by the knowledge of the measurements of objects: in particular, how the sum of the measurements corresponds to a certain arrangement of the objects. The refinement of measurements and the use of the metric system is linked with the study of decimal numbers.

2.5 Probability and statistics

One of the most important of the new subjects beginning to penetrate elementary teaching is probability and statistics. However, even if the will to teach these subjects is fairly general, their introduction at present is still frequently limited to a few experimental areas, especially in the case of probability. We have been able to distinguish various goals in the teaching of probability at the elementary level:

(1) *To provide children with the experience of chance* by presenting them with situations where a determinist model is not appropriate: for example, placing appropriate bets or making correct predictions, even without any analysis. (The words "appropriate" and "correct" mean *satisfying* to the child, that is, fitting in with his statistical experience, which it is absolutely essential to develop.)

(2) *To introduce a useful and precise vocabulary* so that the children may formulate their findings, that is identify events and their "measures". If we wish to simplify the understanding of the model idea, it is important to have two languages, one for the *a posteriori* measures (statistics), the other for the *a priori* measures (probability).

(3) *To construct models of probability*, that is, to clarify relationships between the objects recognized as relevant in statistical experiments, which allow forecasts to be made, and experimental verifications of these forecasts. This is where the use of mathematical tools such as ratios and combinatorics are used.

(4) *To start a more systematic study* by organizing the ideas so as to be able to describe and solve certain categories of problems, while enriching the analytical instrument with concepts such as *events* and *measures*.

It is to be noted that these four goals correspond to four very different levels of abstraction. With the help of this classification, two principal trends can be discerned in the teaching of probability.

A first trend, already fairly widespread, is restricted to the teaching of statistics; in this case the teaching mainly deals with the two first levels, and moreover only achieves part of these first two goals. This teaching is performed with the support of both the information provided by the environment and experiments conceived and carried out by the class.

A second trend, still often at the experimental stage, deals more or less with all four goals. There is therefore the risk of stressing the third goal too soon and of reducing the teaching of probability to a use of arithmetical and combinatory techniques, and of making the simulation without the pupils having actually constructed the models which justify it.

Many didactic materials and situations have been prepared which can be exploited in either way.

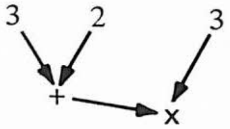
2.6 Influence of computer science

The influence of ways of thinking used in computer science can be noticed to a certain extent in elementary teaching, even if this trend is perhaps less strong at the present moment than might have been expected some years ago. There is no question here of teaching a new subject but of using to advantage certain methods of analysis and presentation by adapting them to the needs of elementary teaching.

A first, very widespread, trend is the use of arrows, boxes, etc., which encourages the understanding of the dynamic aspect of mathematical thought much more easily than a classical arithmetical written sequence. For example the statement:

$$3 \xrightarrow{+2} 5 \xrightarrow{\times 3} 15$$

is simple and immediately understandable. Writing $(3+2) \times 3 = 15$ is not at all on the same level of abstraction as it introduces additional ideas. One method of investigating these notions is the diagram



A second trend is the use of diagrams of the above type to solve and present the solution of problems, and then to generalize from one category of problems in which the terms are similar in nature but different in value. Flow diagrams with or without loops are used with the same end in view and also to describe algorithms, etc.

A third trend consists of using imaginary machines: even if access to a computer is not practicable, children are ready to write a program for an imaginary machine which as a rule can only do elementary computation, and which must be given precise orders to be able to function. The construction and control of such a program compel the children to be very strict in the writing of instructions, and to clarify the properties they use in their choice of elementary instructions.

2.7 Language and logic

Reformers have often been reproached with changing and increasing the vocabulary used in class; let us try and see what the situation is. The real problem in some countries does not concern the quantity, but the function of the vocabulary introduced. Most of the terms are not used to name the mathematical notions themselves, but representations of these notions, or are even used to describe the representations themselves. In other words, these terms have no mathematical function; they have a metamathematical or a perimathematical function.

Let us give an example. There are three names (and surnames, what is more), Venn, Euler, and Caroll, to indicate three slightly different representations of the same mathematical entity (the partition of a set defined by two of its subsets). The partition itself is not discussed; moreover an actual act of mathematization must be carried out in order to recognize this partition in the three cases. Everything takes place as if it were the intention to conceal the subject under discussion from the children. This is a very popular trend and is not new; school vocabulary has always been found to contain more means of indicating or qualifying the significants (the pictorial presentations) than the signified (the concepts). So much so that many concepts cannot be treated in class discussion by the teacher or pupils other than through the intermediary of a pictorial presentation of a concrete situation, which like any presentation, is accompanied by many other things besides the concept itself.

In an attempt to prevent the pupils by-passing the essential (the concept in view) in a concrete situation, a procession of multiple concrete situations follows, which often results in the introduction of as many vocabulary systems as situations, and the need for translation from one situation to the other.

No country has probably been able to escape totally this language perversion, which in other respects is probably no worse than that which has always existed. One of the goals of reformers was and still is the disappearance of this perversion; it is still out of reach and prolonged efforts are still required, as this problem is for many people a question of teacher training, and of mathematical and didactic ability on the part of text-book authors.

In the field of logic, yet another issue for elementary teaching, some practices have been observed deriving from a similar desire to express in concrete terms: for example, creation of an artificial language which is neither the native language nor the mathematical language, resulting in phrases such as "the set of girls wearing non-dresses" (to indicate those wearing skirts or trousers). The current trend is to try and avoid the trap of this artificial language which transforms what is easily understood in the native language into a pseudo-mathematical jargon, in order to preserve that part of it which is most ambiguous, that is, the logical connections. One of the measures suggested is the choice of well-adapted situations (riddles for example) which lend themselves to a twofold description, one in the native language, the other with the help of mathematical apparatus (symbols, diagrams, tableaux, etc) and which allow the children to clarify certain reasonings. But the temptation to make logic an area separated from other activities is still very strong.

3 TEACHING METHODS AND MATERIALS

The teaching of mathematics in the elementary school currently presents a growing diversity of activities devised for pupils, as much in content as in form, while teachers attempt, during each didactic sequence, to adapt – to the best of their ability – the activity chosen with a certain goal in view.

3.1 Choice of situations and materials

The variety of situations presented to pupils within the framework of mathematics, allows them to be shown different aspects of mathematical activity. At each end of the range of these activities one finds, on one hand, those which emphasize the role of mathematics in real life (the application of mathematics), and on the other, those which emphasize the nature of mathematics (creation or rediscovery of mathematics). Of course, many situations combine these two aspects. A balance is on the way to being established at present, but has not been immediately discovered. At the outset of the reform, much more attention was paid to the pure aspect of mathematics, in accordance with the principal concern of the time, which was to introduce the fundamental mathematical concepts into teaching. Certain excesses were no doubt committed in this direction, which caused innovators to be reproached with no longer wishing to show interest in the teaching of useful mathematics to pupils.

But it is only fair to consider all sides of the question. Previous to the reform, the activities devised for pupils were very often confined to the performance of operational techniques and the use of these in problems alleged to have been taken from everyday life, but in fact already half-expressed in mathematical terms and therefore stereotyped. Since then the scope has widened enormously, not only in the creation of mathematical notions but also in the application of mathematics.

Indeed the activities directed along this last path are more than ever rooted in reality, by means of investigations and observation of the environment; they encourage pupils to collect data for themselves and contrive their own problems; this entails that the children must, before treating a situation mathematically, reduce it to mathematical terms, a process founded on a multidisciplinary approach. Once the problem to be solved has been defined, the children can, if the data is too complex, be encouraged to construct a simulation of the situation or look for an approximate result.

At the other end of the scale one finds mathematical games or structured material; they encourage the creation of artificial situations which often depend on a certain number of parameters on which the teacher can play in order to adapt the activity to the children's reactions.

One of the aims of some current research is to characterize situations which may be described precisely in terms of the teacher's strategy corresponding to the development of the pupils' strategy of research. Within this perspective many structured materials are used. Structured material is an effective stimulus to the construction of mathematical notions if it is used to assist mathematization; there is a risk, on the other hand, that structured material may act as a brake if it is used to put a concept in concrete form without its status as a model being clear to the children.

Let us also draw attention to the universal trend of providing more variety in the materials for presentation, by the use of squared paper, abaci, sketches, etc., with the aim either of communicating or of promoting the production of conjectures and investigations.

3.2 Technological resources

Over recent years, certain technological resources for written or pictorial communication have come into general use in the elementary school (like felt-tipped pens, overhead projectors, etc). They make team preparation work of visual documents much easier and provide the whole class with a good quality picture.

The use of some structured materials created with the reform has also become more general. Some of these materials are techniques embodied in material form and are related to abaci (for example, multibase blocks, counting frames, mini-computers), others are investigatory instruments (geoboards, for example). These often very expensive materials have been suggested as being essential to the new styles of teaching, and are for some a great source of profit. Less sophisticated material, made in class as the need arises, would probably be better suited in many cases.

Some countries also use the media (television, films, photographs) to introduce activities to children, or even sometimes to relieve the lack of qualified teachers; but these facilities are used more generally in teacher-training.

If, as might be thought, pocket calculators continue to increase in number, they can no longer be ignored by elementary education; they will have to be combined with both the learning of operational techniques and problem solving: they will provide an opportunity to explore more widely certain areas where overlong numerical computations restrict possibilities at present (e.g. statistics), but research has still to be done in this field where there can be no question of leaving it to machine manufacturers.

3.3 Books and school publications

A most important question is that concerning school text-books and the different publications intended for pupils, as it is mainly through this written material that the renewal in teaching is taking place.

Although, apparently, this question takes a different form in different places, the gamut of possibilities ranges from unrivalled official material to complete freedom of publication for publishers and freedom of choice for teachers, passing through a restricted choice, subject to an accepted conformity; in actual fact, the differences encountered and the current trends are fairly standard from one country to another.

Traditionally, a school text-book has two distinct purposes. On one hand, the text-book suggests outlines of lessons to teachers, who use them adding examples, exercises, motivations, anecdotes, etc. Afterwards, these outlines provide summaries or reminders of the subject for pupils (the text-book only refers to the pupil through the intermediary of the teacher who directs the method of use). On the other hand, the text-book is a collection of exercises on applications of taught material.

A general desire can be noted on the part of text-book authors to appeal more directly to pupils; the presentation is more attractive, the contents are varied; the partition into lesson and exercises is often considered questionable; one finds open situations covering all kinds of areas, supported by photographs, drawings, sketches, graphs, etc. A pupil can organize his work better than before by following his book. Some authors go further in the direction of individualization and suggest, instead of text-books, collections of work-cards, forming a sort of programmed teaching. These collections are accompanied by handbooks with advice for the teachers on their use. These instruments are only really effective if the teacher involves himself in their use by distributing the work, organizing it individually or in groups, and especially by contriving times for collective synthesis, and eventually adding cards himself bearing in mind the children's reactions. Far from diminishing the role of the teacher, on the contrary the new materials require increased availability and skill if they are to be put to judicious use.

Other cards introduce controlled exercises which encourage each child, with his teacher's help, to become aware of his shortcomings and his lack of understanding and orderly thinking, and thus encourage individual teaching and the adaptations necessary to different pupils.

One finds many exercises on these cards with "blanks"; the pupil is invited to complete sentences, diagrams, patterns, etc.; this makes the task of correcting considerably easier, but these exercises are not without risk: (i) their misuse in mathematics and other subjects reduces the pupils' training in organizing and carrying out written recording; (ii) being often based on the principle of multiple-choice questions, their chief disadvantage lies in allowing a correct answer for reasons completely foreign to the basis of the question; (iii) the role of the blank is often ambiguous; it can replace different functions in the same exercise; this ambiguity does not encourage the creation of ideas such as unknowns, variables, constants, etc.

Finally, let us indicate another kind of publication, as yet little known, of a "mathematical club" style. These are collections of problems or open situations entirely for the use of pupils, which they will work on during periods of free activity, and this work will not necessarily find an outlet in a collective class activity.

3.4 Change of methods as regards pupils

Mathematical education at primary school level is currently changing in outlook in a certain number of classes. The child is no longer regarded as a receptacle for knowledge; he is urged to use his intelligence much more. He finds more sources of motivation during his activities (the desire to understand, the pleasure of discovery).

Mathematics may be considered according to various kinds of diversity: (i) diversity of situations which allow the child to exercise his intelligence and skills in the areas he finds interesting; (ii) diversity of work methods which allow him successively to carry out his work at his own pace, to take part in a group within which the activities are shared out and where the findings are discussed, to represent his group in a collective discussion where he must justify the results not only to the teacher but especially to his companions; (iii) diversity of activities linked with mathematics (drawings, manipulations, construction of material, investigations, collection of data on the environment).

Unfortunately, too many classes still exist in which the pupil's life is not organized in this way, but where, with unchanged methods, the work allows the pupils no initiative and still consists essentially in the repetition of similar type exercises. In extreme cases, content change without method change leads to a lower standard of teaching; in fact, if, for example, the repeated chanting of multiplication tables was an ineffectual memorizing process, it was aimed however at acquiring a useful skill, and accepted as such by the children; whereas the chanting of a definition of the intersection of two sets is unprofitable from the point of view of skill, and unnerves the children who do not understand what is expected of them.

3.5 Change of methods as regards teachers

Teaching methods were traditionally based on a progression, separating each learning situation into a certain number of elementary difficulties, which were taught one after the other and controlled by appropriate exercises.

This system was based on the fundamental idea that there is an established body of knowledge and skills to be transmitted. This idea can be adapted to active methods but it necessarily restricts them, and the words "child creativity" have little meaning within such a framework. It is in terms of how they conform to his model of knowledge and skills that the teacher interprets the children's reactions and confirms or rejects their productions, thus creating the confusion between the logical necessity (a statement is true or false according to whether it resists or not the arguments raised against it, and whether the arguments raised in its defence convince class companions) and the necessity of authority (the teacher's approval). Conversely, teaching based on the idea that mathematics is a creation by the class must use more differentiated methods, providing within each topic times for activity, for the formulation of findings or conjectures, and for their confirmation. In the third stage, the teacher's role is especially delicate; he must remain neutral and lead the discussion without showing his feelings or his ideas. These methods are much more arduous for the teacher and also for the pupils, for they must endure periods of frustration and lack of confidence when their early ideas fail; this is the point where the teacher must judge whether he may help and how he may do so.

Such methods are not at all widely used and are still associated with the area of didactic research. They demand special abilities on the part of the teacher. In the eyes of many people they seem costly both in time and energy; they can only be effective if they are constantly used. But the results obtained in this way deserve great attention.

Most teachers, with a greater or lesser degree of success, attempt to change their methods; within a framework which they themselves define and with the skills and techniques they offer the children, they allow the latter much more initiative on the whole, while attempting to avoid the dangers of the two conflicting reefs of dogmatism and *laissez-faire*.

It would, however, be a mistake to believe that a change in mathematical content automatically determines a change in attitude to knowledge, and a change in methods.

3.6 Assessment

The assessment of pupils' work is one of the most difficult points raised by the reform. Previously, it was enough to supervise, through a number of limited points, specified knowledge and skills (operational techniques and certain types of problems), and even if this assessment was fairly arbitrary, it seemed sufficient. This type of control is now no longer suitable as the assessment should also include other capabilities which are more difficult to define in terms of performance. The current situation reveals that a certain perplexity is felt by all when faced with the problem of assessment. One trend, however, can be discerned, which consists in arranging exercises during each activity which provide the teacher with an opportunity of assessing the efficiency of his teaching, so that he may adapt the sequel according to the difficulties he has noticed. These exercises are usually an integral part of the activity and often lead to compulsory productions by the children during this activity; they do not therefore take the form of a check-point in the eyes of the children. On the other hand, individual checking of performance is still being carried out regularly. Very often diagnosing failure is more difficult than before, as there are few well-established criteria.

3.7 Slow and handicapped children

Slow children have always been the despair of their teachers, who did not succeed in helping them maintain the same level as the others. It seems that the new methods of learning computational algorithms, the wider variety of activities and the more suitably adapted pace of the latter, and the mutual aid of their companions help some slow-learning children to overcome emotional blockages to their progress.

Much progress remains to be made in this field where emotional questions play an important role. But the results obtained lead us to think that it is better not to separate slow-learners from the others (no more than the fast-learners): the results of such a separation are generally catastrophic on the psychological plane and moreover unfavourable to collective creation; indeed, for this creation to take place, it is not necessarily the work of each individual, but on the other hand some children must be capable of it, while the others must be ready at that point to understand it, accept it, and take it back as their own.

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Physically or mentally handicapped children are generally taken into specialist institutions, where the educators have very little training in mathematics. Some of the experiments carried out in an attempt to adapt the new approach to arithmetic in this field are very promising (for example, the use of diagrams with arrows and mini-computers with slightly defective children, or the manipulation of sets and their representations for teaching language to deaf and dumb children).

More intensive research in these fields would be desirable, for besides an obvious humanitarian concern it can provide us with much information on the processes of learning in general.

3.8 Mathematics and language

Previously, computation was very dependent on an adequate mastery of the written language, as the terms of a problem were written down and an important part of the solution required a written recording.

Now learning the written language is often a slower process; this encourages the appearance in the first class of a written mathematical language necessary for the activities. Moreover, the methods of classification, organizing information, and substitution, developed in mathematics, simplify the learning of the written language in spelling as well as in syntax. There is therefore a reverse trend in the relationship between mathematics and the written language.

The different levels of the oral language are used much more often than previously in mathematics. Everyday language and gestures are used, if necessary, to exploit a situation; what the children produce is accepted just as it is, if it is understood, so that the rhythm of the discussion is not broken; if necessary, the teacher demands a clearer reformulation or helps the child to do this, etc. It may be said that there is a trend to minimize the handicap of an inadequate grasp of language which burdens children of a deprived socio-cultural background.

Some countries where mathematical activities take place in a language which is not the native language present special problems which have not yet been studied in depth; research on this subject is still in progress.

4 TEACHERS

In elementary teaching much more than at other levels of teaching, the role of the teacher is fundamental. A reform, whatever it may be, can only succeed if the teachers understand its aims, are masters of the new content, and are capable of modifying their teaching methods as a result. Naturally, they cannot all be expected to modify their behaviour in the desired way from one day to the next. Their reactions are widely different, enthusiasm for some, willingness and effort for most, reticence, even refusal, for a few. We shall analyse these reactions in what follows.

4.1 Freedom of teachers in their teaching

It must be remembered that the administrative and pedagogic organization of elementary education varies widely from country to country, ranging from a very centralized organization, where the teachers receive strict directives, to a decentralized organization, where the teachers accept a local educational curriculum which they themselves have often helped to define.

The reform of mathematics teaching generally has the effect, in different cases, of increasing the inequality in teaching between classes, by increasing the freedom of the teachers. This freedom has several aspects; first, a technical aspect: the increased participation of the pupils requires that the teachers be able to develop a strategy which is not entirely defined in advance, so that they can adapt it to the reactions of the children. Next, a sociological aspect: there can be no question of imposing a change in many countries, because of the insufficiently developed ways of providing information for teachers. Decisions for reform provide the most enthusiastic and best-prepared teachers with an opportunity to teach as they wish; the others, caught unawares, see this freedom change to uncertainty; prepared to change their teaching, they do not know how to go about it. For them, the role of publications is fundamental, but they discover the multiplicity of viewpoints concerning content, methods and goals. This uncertainty is contrary to their way of thinking, and forces them to question their very role as educators. According to the way it is applied, the support and the resources they receive, their reactions are very different: progressive adaptation or complete refusal.

4.2 Teachers in the teaching situation

Most teachers are aware of the need for change in their way of teaching. Many are alarmed by the extent of this change. Some can imagine only one way of approaching the class, and thus produce what is indeed a caricature of the method chosen. Experience shows that a teacher can only really change his teaching if he has himself the opportunity to experiment with new patterns of behaviour and to note the reactions of the pupils to this change. The conditions which favour this experiment are grouped very differently according to the different countries. It is important that a supporting body should exist, where the teachers can put forward their ideas and their problems, even if they do not receive a reply which may satisfy immediately (knowing that one is not alone in finding a question perplexing is both morally comforting and an incentive to seek an answer). Periodicals can partly fill this role, but they are not enough; nor are they the ideal instrument for judging materials or text-books, for it is important to judge how the material is used, and not the material itself. Collective thinking is therefore desirable. Another thing which frequently puts a brake on the teacher's change of attitude is his feeling (unfortunately often justified) that he does not have a sufficiently thorough knowledge of the material he is teaching.

He is afraid of not being able to pick out the productive ideas in the pupils' suggestions, and of allowing them to turn into a blind alley, without being able to intervene in a useful way, of making them waste time, and of discouraging them. Many teachers are thus afraid of not having the necessary capabilities, which places them in a very uneasy moral position, made even worse by the often malicious reactions of public and press.

4.3 Polyvalency of the teachers

Teachers in elementary teaching mostly undertake the whole of this teaching themselves; they are not mathematical specialists – which means that their mathematical knowledge is not very great, and that many prefer to devote their time and efforts to other subjects. In many countries the reform in mathematics teaching has been accompanied by teaching reforms in other subjects. Even if the various reforms can give each other support, because they have a common educative aim, it is none the less true that they place a burden of extra work on the teachers and enforce a dispersion of their efforts, which accordingly slows down the development in mathematics. Some countries have specialist teachers at the elementary level; in many others, the problem of specialization is raised but no trend stands out clearly, and the solution will perhaps lie in a partial specialization, progressing with the age of the children.

4.4 Teacher training

Most teachers receive an initial training in which the level of mathematics is not high. In many cases not only the teachers but also their trainers have had to redevelop their ideas towards the new content, which at first led to an emphasis on content (mainly from the point of view of structure) much more than on methods.

The current trend is to bring theoretical training in content and practical preparation for the profession closer together, linking them by an epistemological and didactical analysis. This development is only taking place slowly, as the training institutions are often very lethargic, and their burdens have increased with the in-service teacher-training but without any increase in their resources as a result (and the trainers have therefore no chance to think sufficiently about their work).

Usually the teachers at the pre-elementary level have no special mathematical training and do not feel directly concerned with it in their teaching, but a few wish to find out about the developments in elementary teaching.

5 SOCIOLOGICAL COMPONENTS

When reforms in mathematics teaching at school level became a matter of discussion nearly everywhere in the world, about fifteen years ago, the public and elementary school teachers did not feel concerned. Elementary schools formed a closed world in which the responsible persons were usually not competent in mathematics, and where the developments were mainly guided by the social sciences, especially the educational sciences – which are little concerned with the teaching of arithmetic. The innovations then in progress were totally misunderstood, and the shock caused by the sweeping changes that resulted explains the passionate character of many reactions.

5.1 Projects

The first innovation was the introduction of professional mathematicians and mathematical specialists into the cloistered domain of elementary teaching, when work parties were set up in the different research and training projects and in the official commissions (although these contacts concerned only a few individuals). The reform movement was set in motion at the instigation and according to the ideas of a fairly small number of people, and the few research projects in progress on the teaching of elementary mathematics served as a work basis for the commissions set up in many countries, without any original experiments being developed at the same time. One might have had the illusion of a vast international consensus, whereas it was rather a freak echo reverberating around a fairly small range of suggestions. Since then, view-points have varied greatly and the following trends can be discerned: (i) *a structural tendency* (important at the outset) which emphasizes the teaching of mathematical structures, with the aim of treating the traditional subjects in a new way; (ii) *an arithmetical tendency*, characterized by the introduction from the earliest schooldays of the language of sets, and by an approach to arithmetic which makes it much more an autonomous unit of skill than a tool used in the quantitative study of the environment; (iii) *an empirical tendency*, in which mathematics teaching is done through varied activities in the areas of measurement, geometry, functions etc., concentrating on the didactic approach rather than on a vertical, logically arranged organization.

5.2 Transformations

The shock needed to set the reform in motion was produced much more by the change in content, because of its spectacular aspect, than by the change in methods (it is significant that articles in the main newspapers deal almost exclusively with content and "teaching of new mathematics" is spoken of rather than "new teaching of mathematics").

Change in methods requires that teachers should have both a long psychological and technical preparation. But to carry out this preparation certain resources are necessary which initiators of reform have long demanded in vain. The authorities in charge understand more easily the need to provide teachers with a training in content than a training in methods.

In many countries, establishing the teaching of new content, informing the public and training teachers had all to be set in motion at the same time. The role of publications became therefore fairly important, as much those intended for the eyes of the public (the quality of which is often suspect) as those intended for the pupils and teachers.

It is through the intermediary of text-books for the pupils and accompanying handbooks for the teachers that the large majority of teachers come in contact with the innovations which it is their duty to teach. The early text-books, often hurriedly produced and relying only incidentally on the author's actual experience, helped to emphasize the gap between the initial projects and their implementation. In the countries where the reform promoters themselves produce material intended for teachers, the results, although better, are not perfect, as this material seems to the teachers too far removed from their way of thinking for them to use correctly, and they often prefer other more traditional text-books. In any case, the teachers have to make a great effort to adapt, and generally do so without opposition.

Parents' reactions range from interest and curiosity at the outset, to anxiety and considerable opposition in the face of the early achievements. The emotions which were aroused are now, on the whole, soothed. But how much resignation is hidden by this apparent lack of interest?

5.3 Difficulties

One of the reasons for the divergences noted between the early endeavours and their more general use is probably that the method changes recommended are too often justified by general pedagogic considerations rather than by the special requirements of mathematics.

So a gap is to be seen between the pedagogic wishes of the teachers, which are: to pass on knowledge, to correct through their teaching socio-cultural disparities and to encourage maturity in the children; and the teachers' everyday practices, where by concentrating their efforts on technical skills they reduce the cultural role of mathematics.

Evidently these questions also preoccupy the parents, who generally tend to distrust the cultural aspect and prefer that learning deal only with basic techniques.

This antagonism between technical skills and mathematical culture, expressed by many people, is an erroneous idea; many experiments show that the learning of the former is made much easier by the construction of the latter.

An improvement in the situation can only arise through a change in attitude towards mathematics. As long as teachers do not have the opportunity, during their training, of practising mathematics themselves in a constructive way, and of seeing the same attitude of discovery in children, they will be tempted to look on mathematics as a fine building, up whose steps the children must be made to climb one at a time.

By making a distinction between the aspect of knowledge and that of mathematical language, we have sanctioned the idea that the change was confined to modifying a manner of speech. We have even seen books for the use of elementary teachers, based on the plan: "one used to say . . . now one says . . .".

The plans for help in the classroom, proposed by the reform commissions at the same time as the new directives, have rarely been implemented by those in charge of elementary teaching. The use of television as a means of continued training for teachers is disappointing if there is no structure allowing teachers to work out the contents of a programme as a team, and to communicate with the producers.

But the inadequacy of the resources does not explain everything. We lack information concerning the precise conditions of an effective training for teachers (often the proposed curricula have been too much influenced by the structural orientation, and certain modifications have had to be made). Research on this subject is in progress.

Recent years have seen the establishment of "think tanks", bringing together people of very different training and ability; but their existence seems to be under scrutiny at present in certain countries.

6 RESEARCH AND PROBLEMS

One of the main difficulties encountered in speaking of the reform in mathematics teaching at all levels, and perhaps even more at the elementary teaching level, is the lack of any system of ideas which might be used as a reference. The most widely-spread ideas on elementary teaching of mathematics in most countries are based on learning by conditioning (behaviourism or neo-behaviourism). However, the reforms have been introduced precisely to fight against this method of learning, which seemed to have a special link with the learning of arithmetic.

When psychologists' information on the child's operational stages coincided with the endeavour to reorganize mathematics into large structures, it seemed an opportune moment for the creation of a new didactics of arithmetic at the elementary level, based on the learning of structures and integrated with the wave of unification in scientific thought.

Research began in this direction, and the results of such research encouraged the start of the reform. Subsequently this research was disseminated, for the most part, according to the principle of conditioning, applied this time to the teachers.

6.1 Current research

The spread of the reform has resulted in research being multiplied and diversified with generally an increased concern for scientific exactitude, often in an empirical way, and less often within the framework of an embryonic theory. Thus we have been able to see:

- Investigations or assessments modelled on the experimental sciences, requiring considerable resources, but the results of which are often to be treated with caution, since, although the design and the processing of data are scientific, the selection and collection of these data are not always done in a satisfactory way (much of this research has been conducted without the participation of mathematicians).
- With a view to helping teachers solve the practical problems of teaching, numerous pointillist studies and a spread of documents adapted to short teaching sessions, emphasizing learning by conditioning.
- A large number of studies in restricted fields in which the display, the description and the transmission of pedagogical methods seem to be more limited. This leads to a fragmentation of knowledge, which is the very opposite of what was envisaged. These researches are based on the idea that all knowledge can be reorganized in the framework of large structures, this

reorganization being, however, only interesting if it encompasses enough things and is carried out at a sufficiently high level.

There are attempts at present to establish the didactics of mathematics as a fundamental autonomous science which must create both its objects of study and its methods; the latter not being reduced simply to a transfer of the methods of the experimental sciences, which are often ill-suited in the present state of knowledge. This will probably cover mainly the field of elementary mathematics teaching as, on the one hand, we have here the most experience (psychological and other) and the problems are simpler and more fundamental; on the other hand, it is here that there is most urgent need for progress.

6.2 Some problems

Recent years have seen the appearance and formulation of many problems. Amongst the most important and widespread let us mention:

- Research on situations favouring fundamental processes of learning (arithmetic and others); the study of conditions necessary for reproducibility. (In particular, in a teaching sequence, what are the choices open to the children and the choices open to the teacher? Can criteria of choice be established so that the teacher may construct his strategy?)
- Research on the operating conditions of communication necessary to prevent knowledge becoming a theory detached from the problems which gave it birth (at the classroom level as well as at teacher-training level). In other words, research into functional and not structural methods of communicating mathematical or didactical knowledge.
- Research into the role of analogy in mathematics learning; is analogy a method of discovering structures, or already a competence recognised as such by the person learning? (Research not only into the pupil's ability to express in mathematical terms but also his ability to theorize.)
- Study of the importance of a child's personal discovery as he acquires knowledge (the importance depending on the nature of that knowledge), and its effects on teaching methods.
- Research into the difficulties of slow-learning or handicapped children. Has the medical character of certain handicaps, which has been asserted, been proved? How can a diagnosis be made, and what treatment can be provided which is not pedagogical but medical (in the widest sense)?
- Elaboration of didactical theories taking the topological aspects of content into account; especially for the introduction of decimals; or in geometry, perhaps allowing the latter to produce functional learning processes (based on the adaptation and mathematization of concepts).

The study of these problems raises many political and institutional issues which are not peculiar to research on mathematics teaching, but which, as a result, make it difficult to study in a total and controlled manner the whole of the innovation process, ranging from experiments with pupils to the training of teachers and application in the classroom.

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PANEL

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Chapter II

Mathematics education at the first level in post-elementary and secondary schools

A. Z. Krygowska

INTRODUCTION

This chapter is concerned with mathematics education for the age level 10 to 16. It consists of two parts. Section 1 constitutes the first part, which deals with the overall background and sources of the current developments in the field of mathematics education. Sections 2–6 constitute the second part, which deals with trends and problems in important areas related to the level under consideration: goals and objectives (section 2); content – and its organization (section 3); methods and assessment (section 4); teacher education (section 5); research perspectives (section 6).

1 GENERAL SOURCES OF CURRENT TRENDS AND PROBLEMS

1.1 Some important changes

In many countries, post-elementary school education has become compulsory for large numbers of children up to the ages of 14 to 15–16 years. Within this education mathematics generally holds an important position as a compulsory subject for all pupils up to the end of their school life, or at least as an optional subject during their final years at school.

The invasion of the post-elementary schools by these large numbers of children naturally demands a great effort on the part of society, with a view to providing the material conditions and the qualified teaching staff for education at this level. But a great effort is also needed to formulate a conception of "mathematics for all" and the final aims of this teaching, as regards the individual and society itself. The lack of such a conception is reflected in the fluctuating opinions concerning the position of mathematics teaching in general and in professional education at post-elementary level. On one hand, particular importance is attributed to this teaching in accordance with the role of mathematics in the development of science and technology, and the role of its teaching in the process of intellectualizing modern man's attitudes. On the other hand, there is opposition to the fetishism of mathematics in school education for "all", and this opposition reflects the fundamental criticism of the scientific and encyclopaedic approach of this education which (according to this opinion) disturbs the balance in the development of general culture in young people, and impedes their preparation for active life.

From this point of view, the report of the National Advisory Committee on Mathematical Education in the United States presents a particularly interesting example of the analysis of such fluctuations and their sources (NACOME, 1975). This report establishes the following facts. – The current situation of mathematics in the United States is disturbed by internal and external difficulties. – Public opinion does not now accord to the mathematical, scientific and technical disciplines the same privileged position they held after the turmoil provoked by the launching of the first Sputnik. – Young people's interests are directed towards other subjects and the popularity of mathematics, and consequently its financial support, is neither as strong nor as significant as previously.

Removing the cloak of mystery from the role of mathematics is "probably" (as the authors of the NACOME report say) justified. But unfortunately this role has been undervalued in an exaggerated and disquieting way. It is therefore necessary to reaffirm the importance of mathematical education for society and the individual. Such a reaffirmation demands serious answers to two basic questions: "Why mathematics for all?" and "Which mathematics for all?".

Naturally such fluctuations do not occur everywhere in the same way. In some countries (for example, developing countries) acceleration of the training of national scientific and technological staff is treated as an indispensable condition of progress and political and economic independence. Secondary school in such countries being practically available to only an elite of pupils who will pursue a more or less long course of studies, the question of "mathematics for all" still only concerns the ages from 6 to 11–12 years. But in spite of the differences due to the varying developments of societies, similar problems arise everywhere when a certain level on the educational scale becomes compulsory for every child in the country. So, for instance, in some developing countries, one feels the urgent need for the adaptation of mathematics in the elementary school to the true needs and potential of the mass of children whose school education finishes at this level. A similar problem concerns the first stage of the secondary school in the countries where education at this level has already become the education of the majority. If the mathematics taught in the lower stage of secondary school is conceived as the lowest floor of a rigorous global edifice of elementary mathematics, where certain intuitively simple and useful pieces of knowledge can appear, for theoretical reasons, only on the upper floors, the mathematical education of the majority of pupils (who are not going to climb to these floors) is curtailed. In this situation, this majority only derive from their mathematical education "little knowledge and few skills which will be useful for their professional training or their future work" (French Report).

In some countries, the teaching of various mathematics courses, according to the choices of the pupils, begins before the age of 16. The problems raised here do not concern the mass of young people as much as the unitary mathematics education for all at the lower level. But these problems are not and cannot be divorced from the problems of the mathematical education of the masses, this inexhaustible source of the intellectual layers of every country.

A search is therefore being made to find ways of organizing mathematical education at the lower stage, so as to make it reasonable and useful to each child, as much to the child whose mathematical education will finish at this level as to the one who will follow the mathematics course to the higher level. But what meaning should be given to the expression "useful for all"? There is a need for a clearer idea of the common mathematical culture, a minimum culture which could and should be developed in all young people, becoming a deeply integrated part of their general culture, without primitive pragmatism or fetishism of mathematics taught for its own sake.

The education of large school populations demands large numbers of qualified teachers, and mathematics teaching at the level of the middle-year classes demands special mathematical and pedagogic qualifications, for it is a distinct stage in the totality of mathematical education. This is the stage where the pupil's mathematical world is not only enriched but also adopts a more methodological structure. The transition from what was implicit in the mental experiences of the primary school child, to something explicit in the thought of adolescents at the age of 15–16, requires not only an adequate theoretical understanding of the course material but also the application of discriminating methods. In his role as scientific guide and as an educator sensitive to the difficulties of the transition stage from childhood to adolescence, the teacher should have a broad, deep-seated mathematical and pedagogic culture at his disposal. But it is obvious that teachers cannot be recruited in a body from a restricted elite. On the contrary, it is essentially the large numbers of average teachers who shape the common mathematical culture of society. To forget this reality while setting up any reform in mathematics education on a national scale, would be to run a great risk as regards the education of a whole generation.

The invasion of schools at post-elementary level by these large numbers of children, and the resulting change in the social structure of the middle-year classes in general education, is therefore one of the main sources of the problems analysed in this chapter.

1.2 Criticism of some reforms in the "first wave"

During the last ten years the reforms aiming to modernize school mathematical education have undergone preliminary analyses and assessments. These assessments were not always made in competent and unprejudiced circles. But they did also provide serious and profound criticisms, which have a considerable influence on the current trends and research. It would be impossible to reproduce here all sides of this earnest discussion. We shall therefore only note the main targets of the criticism concerning the so-called "Bourbakist" orientation of certain reforms, namely: (a) the fetishism of the set theoretical mode of thought, (b) the abstractions which are unproductive because they are unjustified in their applications and often wrongly embodied in concrete situations, (c) the pseudo-erudite language burdened with symbols and terminology, (d) the fetishism of the axiomatic method, (e) the fetishism of rigour, which in the reality of the classroom has become useless pedantry, (f) the neglect of physical reality as a source of mathematical ideas, and in particular the neglect of physical space as a source of geometry, (g) the neglect of an over-all vision based on spatial awareness, to the advantage of the algorithmic thought of formal algebra (Freudenthal, 1973; Kline, 1973; Thom, 1971, 1973).

This attack, at the peak of mathematics, directed at the principles of the "Bourbakist" orientated reform, has been strengthened by the criticism of the implementation of this idea in the classroom situation. "Unfortunately the innovations have not fully achieved the euphoric hopes of the year 1960" states the NACOME report (1975). According to some observers, the so-called "new" mathematics has proved neither easy nor interesting for many pupils, and has continued to be the means of selection favouring children more conceptually developed because of their home background (Jaulin, 1974). The new methods, often ingenious, involve in practice misconceptions and erroneous concrete expressions of mathematical abstractions (Freudenthal, 1973; Adda, 1975). In the light of certain assessments, pupils' performances arouse anxiety. The application of methods activating the pupils' thought in open situations is rendered difficult by the deductive structure of the lesson. The level of abstraction eliminates from the pupils' experience many simple problems, accessible to them in free research. So-called "new" mathematics is isolated from other teaching subjects.

To each of these criticisms equally energetic replies are given. At the peak of mathematics, the so-called "Bourbakist" orientation of the reform is defended from the standpoint of the scientific foundation of mathematical education (Dieudonné, 1973). The criticism is accused of not being objective, of generalizing the pathology of the reform, of attacking the theoretical model without knowing what actually happens in teaching (Begle, 1974). The means of assessment are also suspect as being too superficial to be able to measure the complicated and multidimensional phenomena of mathematical performance. In opposition to the polarization expressed by the dichotomies: traditional or new mathematics, skills or comprehension, concrete or abstract, induction or deduction, intuition or formalization, mastery of structures or practice of applications .... stands the principle of unification as the controlling idea of the reforms of the "second wave".

The criticism of the most radical reforms of the "first wave", in spite of having given rise occasionally to incompetent and demagogic demonstrations, has been useful to the progress of mathematical education. It has encouraged people to define their positions clearly and it has revealed the danger of illusions concerning the reality of the classroom. It has shown the importance of a clear conception of mathematics education "for all", and the need for a pedagogy which is not confined to the construction of a modern curriculum of elementary mathematics. It has given rise to new problems and new trends.

1.3 Society's influence

Cultural, structural and economic changes exercise an ever-increasing influence on school education in general, and on mathematical education in particular (unemployment and competition, overproduction of intellectuals in some countries and their shortage in others, mechanization and standardization of production, the growing role of computers, changes in family life, standardization of the measurement system throughout the world, etc). Price reduction in calculators and the production of didactic resources on an industrial scale open up new horizons in mass education on one hand, but on the other, arouse some anxiety concerning the pressure of commercial interests on teaching.

In many developed countries – having a long tradition of publicly managed school education – society has become knowledgeable and active enough, when faced with social problems, to be able and to wish to influence and assess the working of its educational institutions. Here the school is under pressure from public opinion, from demands and criticisms coming from different backgrounds. The fluctuations in mathematics education caused by this pressure have sometimes proved unfavourable to progress, but they have also indicated the need for an analysis of the reasons for these fluctuations, and as a result, the need to rethink certain conceptions. In some developing countries, the lack of a tradition of compulsory schooling for all children on the one hand, and the firm traditions of family life on the other, as well as the economic situation of large layers of society, still create obstacles to the development of school education for the masses, not only at secondary level but also at elementary level. The prejudices concerning the mathematical education of girls are sometimes revealed in developed countries also.

But if society exercises an influence on the progress of mathematics education for the masses, this education in turn moulds the average mathematical culture of society, and develops the understanding of the importance of this education. The awareness of this feed-back with the need for the formulation of adequate strategies during the reforms has today become one of the sources of the current problems and trends in the teaching of mathematics. Proof of this awareness is to be found in the conferences, discussions and recommendations of the recent large congresses held in Latin America (Caracas, 1975), Asia (Bharwari, 1975), and Africa (Rabat, 1976). We need only compare the spirit of the Rabat congress with that of the regional seminar held in Cairo (Unesco, 1969) to understand the development of the idea expressed in the Rabat recommendation: "The congress recommends that the content of curricula should be at once international, pan-African, and national". "The congress is against all importation of school or university structures by the African countries, and suggests the formulation of teaching systems and curricula in accordance with our concern for our development".

1.4 School structures

The variety of school structures at the level under consideration in this chapter reflects, on one hand, the diversity of conceptions of education, and on the other, is a source of different trends and problems in different countries. There are essentially two decisive aspects: the transition from unitary and compulsory universal education to diversified education, and the amount of freedom in teaching as regards the administrative system.

As examples of conflicting structures according to these two aspects, we can mention school education in the Union of Soviet Socialist Republics on the one hand, and in the United Kingdom on the other. In the U.S.S.R. a diversification into four streams takes place after eight years of education in a unitary, compulsory system for all children. The mathematics curricula in all these streams are practically the same, in this way therefore all normal adolescents in the U.S.S.R. receive a compulsory, almost unitary education for at least ten of their school years. In the United Kingdom the diversification begins early (at the age of 11–12 years). Of the many courses in school education at this level, two streams lead to examinations taken at the age of 15–16, by the pupils wishing to obtain one of the two types of secondary school certificate. In the U.S.S.R. all school education is administered centrally at Federal Republic level; in the United Kingdom on the other hand, teaching is unrestrained, decisions on course content being left to the teachers or to a development project (many experiments and different projects) and directed towards the examinations intended only for certain groups of pupils (although these constitute the great majority).

It is clear that the problems raised in mathematical education in such different systems are to a great extent themselves different. But certain trends reflect the conflicting aspects of the same universal problems. For example, in the countries where the division of pupils at the start of their post-elementary education, made according to the standard of their abilities, still remains one of the fundamentals of the system, the teaching of "mixed ability" classes in under investigation. On the other hand, in countries where the principle of teaching mixed ability classes, following the unitary curricula for all pupils, is achieved successfully at the secondary level, the problem of pupils who are conspicuous very early by their abilities and the interest they show in mathematics demands research into methods of individualized teaching within the framework of the system. Thus the same difficulties and problems are revealed when they are analysed within the various school structures.

2 GOALS AND OBJECTIVES

2.1 General trends

Over the last ten years, a great effort has been made to define the final goals and objectives of teaching mathematics. The interest shown in this problem by various groups of people becomes understandable in the light of the new social status of the secondary school. Society expects effective working results from its educational institutions. But to evaluate these results there must be a clear awareness of what is expected. It is felt that there is a need for clarification in this field, even in countries where the tradition of pragmatism and liberty in education is opposed to theoretical analyses and the conditioning of teaching by the aims imposed by the country.

In fact, it still happens that the goals and objectives of mathematics education at secondary school level are defined according to the goals and objectives of higher education. The new social status of the secondary school as the school where the mathematical education for the majority of pupils stops, prevents us from accepting such a principle.

Mention must be made of work directed towards a theoretical analysis of the formative potential of mathematics teaching (Lenné, 1969) on the one hand, and on the other, towards a clarification of the goals of such an education in the school system as set out for example in Stoliar's book (1969), or in the Hungarian curriculum plan (*Orszagos Pedagogical Intezet*, 1975), or in the Japanese curriculum (Japanese Report), etc. These goals and objectives concern the qualifications, the attainment of which by those who pass through post-elementary education should be ensured by one of the following variants: (a) within the general education system, (b) in classes with diversified curricula which are directed towards future long-term studies, (c) in secondary level streams which are directed immediately towards some given profession.

Prevailing opinions converge towards an expression of the general aims of mathematical education for all in the following terms: (a) to develop mental activities and to intellectualize attitudes, by putting the pupils in contact, active contact, with the rudiments of a science which as a subject of study is particularly favourable to such a development, (b) to provide the pupils with knowledge and skills, and with a conceptual tool and the ability to use it, all of which are necessary for active and intelligent participation in contemporary society, (c) to ensure that those who continue their studies have an essential basis as regards mathematical preparation on the one hand, and the ability to study mathematics on the other. These formulations are expressed in concrete terms by such specifications as, for example, "the development of logical reasoning", "the development of spatial awareness", "initiation in the axiomatic method", etc., and by curricula arranged and expressed in different forms, also in the shape of lists of operational objectives.

Tradition dictated that school work was directed according to the following reasoning: sophisticated mathematics, as a method of thought and as a universal language, is not available to the average pupil. Mathematics learning for these pupils should therefore be concentrated on abilities which have been made automatic, on memorizing content, and on skill in solving stereotyped problems according to assimilated patterns. This reasoning is false. If it were true that it is impossible to develop in the majority of children, through the learning of mathematics, their intelligence and intellectual acitivity, there would be no sense in burdening their minds with knowledge and skills acquired with so much effort; for not only would they largely never use what they had acquired, but the effort of memorizing without active participation in the actual building of the knowledge could hinder the development of their general intelligence. So, if the aforementioned premise were true, the necessary conclusion would be: to eliminate post-elementary mathematics from the education of the masses, and to preserve it only for future users who are more able in mathematics. But opinions are developing in the opposite direction. Confidence in the potential of adolescents is springing up parallel to the improvement in methods and means of learning.

General education, including mathematical education, is treated as an essential factor in the polyvalent preparation for a full life. The current innovations are to be seen in the curricula of professional post-elementary schools, where the role of general subjects, mathematics among others, is becoming more and more significant (*Bureau International de l'Education, 1974)*. The fairly wide scientific basis and the intellectual attitudes developed along with the learning of this basis provide professional preparation with an adaptability to rapid changes, which is particularly necessary in the contemporary technological world.

From this point of view, mathematics education is treated in socialist countries as an important factor in the polytechnic education of the masses. Mathematics education "for all", from the point of view of the principle of the polytechnic approach, should (a) make each pupil aware that mathematical research is not about particular phenomena but about their abstract models, and that for this very reason, thanks to its character as a science of models, it is useful because of its wide application, and (b) initiate each pupil in the construction of such models, and their use in the solution of problems adapted to his level, having a practical, technical or economic application (Gnedenko, 1973). In this context, the "usefulness" of learning mathematics is understood in a very different sense from narrow pragmatism.

Attempts to achieve a balance between the technical aspects of professional preparation with the conceptual aspects of scientific subjects are met on the other hand by attempts to professionalize, to a certain extent, general education (career oriented curricula), which also concerns mathematics. These are still very vague proposals, but which reflect an underlying idea of the integration of general and professional education (NACOME, 1975).

2.2 Operational objectives

There is an essential difference between the concept of an objective defined precisely by such expressions as "to know the theorem . . . ", "to be able to solve the equation of the type . . . ." etc., and the concept of goals defined in a more vague way by such expressions as "to initiate", "to develop", "to familiarize" etc. The key-expressions "to develop an open attitude to problems", "to initiate in the process of mathematization" etc., only indicate guide-lines for the process of teaching and the meeting points of various avenues of learning, which are never finally reached. These are goals specific to mathematics education, which are being approached the whole time if education sets out to be truly formative. The keystone of mathematical teaching would therefore be the ability to continually guide the achievement of the more direct objectives towards these general goals.

If the teaching process is considered within such a perspective there is the difficult problem of assessing the effects of teaching from all these points of view. It is possible to control and evaluate by well-known methods, attainments, algorithmic abilities, skill in solving standard problems according to previously assimilated patterns, although there are still many difficulties even in this field. Progress concerning "development", "initiation", and "familiarization", etc. does not, on the contrary, correspond to the primitive methods of objective evaluation which are currently available. The drastic attempt to express all attainments by objective measures often reverses the order in the didactic procedure: "to set oneself the goals, and to look for the adequate means of control and evaluation of their achievement". This reversal is expressed in the tendency to set objectives for mathematical education in accordance with the tools of control and evaluation available. Such procedures have been developed especially on the basis of behaviourism. From this point of view, the specific objectives are defined by detailed lists of the simplest performances, which define in the same way the minimum learning curriculum and which can be assessed locally in simple situations by suitable standardized tests.

The NACOME report analyses the effects of the implementation, in a school situation, of such a conception of objectives in the United States. It is observed that very often the effort of teachers and pupils is concentrated on mastering restricted, isolated performances as defined by the lists previously mentioned. The minimum curriculum confined to these performances also leads to minimalization as regards the organization of varied mathematical activities; it does not suggest open situations, free research done by the pupils, etc. The authors of the NACOME report emphasize that the correct learning of "true mathematics" demands that pupils be initiated in heuristic procedures, which cannot be induced by problems broken down *a priori* into a sequence of small exercises. They state ". . . we agree that . . . an overly-specific conception of behavioral objectives is a severe danger to mathematical education and we deplore much of what is current practice in producing more and more lists of low-level, narrow skills as the sole or major objectives of mathematical instruction" (NACOME, 1975, p.55). The report therefore indicates an urgent need for research into means of assessing the development of mental attitudes, accurate thought, heuristic procedures in problem-solving, of all these attainments of intelligence which ought to be considered by mathematical pedagogy as learning's final aims.

We should also note the work concentrating on the construction of lists of operational objectives in some countries where, up to now, mathematics teaching has not been guided by such means, and the research carried out in these countries regarding the influence of such an orientation on the results of learning (e.g. Tourneur et al., 1974; Tourneur, 1975).

The problem of methods and strategies in the orientation of mathematics teaching by the definition of its final aims is still open.

3 CONTENT, ITS ORGANIZATION, CURRICULA

3.1 Common ground – and differences

The common core of the different mathematics curricula taught at the level considered here contains the following topics: real numbers, numerical computation, elements of traditional algebra, affine functions and examples of other functions, equalities and inequalities, systems of simple equations, Euclidean geometry of the plane, Cartesian co-ordinates, geometric vectors, similarities in the plane.

Beyond this common ground, the following subjects are to be found: the elements of logic, algebra of sets, Boolean algebra, elements of modern algebra, classical functions (polynomials, rational functions, logarithms, exponential functions, trigonometry), equations and inequations including systems of higher degree, elements of analysis, combinatorics, probability, "mathematics for life" ("consumer mathematics"), various subjects chosen from the point of view of applications, affine and Euclidean three-dimensional geometry.

3.2 Statistics and probability

Descriptive statistics and probability are beginning to take an important place in the mathematics courses at the level under consideration. It is agreed that the notions of statistics can illustrate certain mathematical concepts and motivate computation. Learning the elements of descriptive statistics can be broken off at the different levels corresponding to the study of numbers, functions and co-ordinates etc.

The position of probability within elementary mathematics, at the educational stage which interests us here, creates on the contrary certain disagreements. According to one idea, there is no question at this stage of learning a theory, but essentially of acquainting the pupils at an early stage with probabilist thought in action, which is different from formal, "determinist" thought. The work carried out during the last ten years has introduced many new ideas into this field, and various projects have been worked out in detail (Engel, 1975, 1976; Engel et al., 1974; Glaymann and Varga, 1973). According to opposing opinion, the theory of probability is a mathematical theory like other mathematical theories, and should be taught as such. It is therefore only suitable at the teaching level beyond the lower stage of secondary school.

This discussion reflects deeper dissensions concerning the very idea of so-called elementary mathematics. But the elements of probability are already treated informally in many countries at the level of primary school and the middle-year classes.

3.3 Combinatorics, finite structures

In conjunction with this trend, the teaching of combinatory analysis also arouses much interest. We also notice the increasing role given, from the pedagogic point of view, to finite structures (e.g. Myx, 1973). According to some experiments, such structures very soon simplify the children's active approach to the world of abstract structures, starting with manipulations. But some doubts are also noted concerning exaggeration in this field, which can develop the pupil's thought unilaterally, and may even hinder the development of the heuristic procedures needed to handle problems which cannot be solved simply by finite enumeration.

3.4 Geometry

The question of geometry teaching still remains open. With regard to the situation described in chapter 3 of volume III of *New Trends in Mathematics Teaching* (Unesco, 1972), we note certain recent endeavours, which appear in different ways but which all aim at a geometry taught without being conditioned by "the system" or developed in different ways before its integration into the system (French Report; IOWO, 1976; *Bericht über IMUK Regionaltagung*, 1974; Vysin, 1975; Papy, 1974). We should mention here the extensive analysis of the problem of geometry teaching carried out by H. Freudenthal (1973).

3.5 Status of general structures

Important differences in content stem from the different ideas concerning the role and place of fundamental mathematical structures in teaching at the level of the middle-year classes (algebraic and topological set structures).

Three points of view may be distinguished: (a) Sets and algebraic structures are treated as subjects appropriate for direct teaching (the algebra of sets, Boolean algebra, elements of group theory, study of vector spaces). (b) There is merely a question of a universal language, worked out on the basis of concepts introduced either intuitively or by definitions generalizing very varied examples. This language facilitates integration of the course and communication. No theorems of group theory, for example, are introduced, but the term "group" is used in the various situations. (c) In principle abundance of terminology is avoided by organizing the teaching in such a way that the structures underlying different situations become more and more familiar to the children during their mathematical activity, without having to define them and give them their explicit names.

3.6 The axiomatic method, deduction

It is usually considered that pupils should be introduced to deduction by local deductions, studied very carefully and progressively from the methodological point of view. According to this opinion, axiom-based global deduction and formal nuances are not within the reach of the majority of pupils in the middle-year classes. We must nevertheless emphasize that in some countries the axiomatic basis of the course is introduced quite soon, even at the start of secondary school, and that the development of elementary mathematics in such a way is defended.

3.7 Integration

The term *integration* is used in the three following intepretations: (a) *internal integration* at the heart of mathematics itself, (b) *external integration* concerning communications with the other teaching subjects, and (c) *vertical integration* concerning the consecutive levels of education.

(a) *Internal integration* is achieved mainly according to one of three models: (i) with general structures as a base; (ii) around problems and applications; and (iii) with the help of "transverse bridges". According to the first model, elementary mathematics is presented in the form of a building, founded on some general structures, growing in height deductively. It is characterized by explicit axioms, by the linear organization of the course through the Socratic method of question and answer (for example, some Belgian text-books). The second model tries to integrate the teaching of the subject on a basis of applications and the solution of problems. Instead of constructing *a priori* the deductive "building" of elementary mathematics, a sequence of situations is chosen, the exploration of which allows a fairly wide curriculum to be covered. Local deductions are included in this process, and the lesson is often organized in the form of a spiral (Castelnuovo and Barra, 1976; IOWO, 1976). Thirdly, if the content is developed following certain vertical lines (geometry, numbers and numerical functions, combinatorics, probability etc), the "transverse bridges" are applied as often as possible between certain points on these lines (demonstration of analogies, generalization of notions defined independently at first on the various lines etc, for example, in the curricula of the U.S.S.R. and Poland).

(b) *External integration*. The attempts at such integration are more significant at primary than at post-elementary level. Some positive results are obtained if one teacher teaches two or three subjects, or if the teaching specialists work as a group (*Académie de Grenoble*, 1976). The lack of co-ordination in curricula, language and definitions may be noted. Pupils have many difficulties in transferring attainments from one teaching subject to another. It may be observed that the set-oriented language of mathematics increases these difficulties if the teachers of the other subjects do not use this language. Difficulties may be noticed in introducing pupils to non-traditional applications; this often stems from the teacher's lack of knowledge of the fields opened up recently by these applications. The introduction of statistics and probability at the teaching level under discussion, and the use of calculators, have widened horizons in the field of computational applications, in programming and in graphical representations. We should note some studies devoted, on the one hand, to the theoretical analysis of applied mathematics (for example, Blechman et al., 1976), and on the other hand, to examples of the applications (Bell, 1967; Castelnuovo and Barra, 1976; Echternach Symposium, 1973; Galion, 1972; Fletcher, 1972; Pollak, 1970).

Some progress concerning external integration has been made on the frontier of mathematics learning and computer science. The use of computers in mathematics teaching is not as yet significant in the teaching of the majority of pupils, but we are already aware of the possible contribution of this field to mathematics teaching. On the other hand, we are also aware that mathematics teaching can and should develop attitudes and abilities which would encourage the development of thinking along the lines of informatics as a precise mode of thought, to be used in the organization and presentation of reality with the help of the type of models computer science provides (*Institut de Recherche*, 1972; Kuntzmann, 1974).

(c) *Vertical Integration*, primary school – secondary school, poses no problems within the unitary school education system up to the ages of 14 to 15–16. Obviously, two or three stages can be distinguished here, but the transition from one stage to the next is simply a transition from one class to the next in the same school. Some difficulties for the child come to light when they move from the class of a single teacher to a class where the various subjects are taught by different teachers. These may be linked with the too rapid change in methods and with disturbing emotional aspects. But the construction of the entire course for the whole school is based on adequate vertical integration. Conversely, if the primary school is isolated from the secondary school (and in extreme circumstances even isolated from the point of view of administration and teachers' associations) the situation is different. It worsens when the planning of the teaching is open, without co-ordination between the successive stages by means of compulsory curricula, and if moving to one of the post-elementary streams is conditioned by selection at the end of primary school (teacher's opinion or test). In certain developing countries, where in spite of secondary schools being officially open to all children, mathematical education stops for the majority at primary level, the problem is even more complicated by the economic and cultural conditions of the population. The élite who go on to secondary school are not identical to the élite from the viewpoint of their mathematical capacities. The same difficulty concerns the transition from the lower to the upper stage of the secondary school in some developed countries.

Vertical integration in mathematics education does not finish therefore with the curricula. It also concerns teaching methods and assessment procedures. In some countries an attempt is being made to achieve such an integration through direct contacts between teachers at various levels. But it may still happen that the secondary school teacher neither knows nor is interested in the teaching which his pupils received previously. In the teachers' initial and final training, the problems of vertical integration should be considered to a far greater extent than hitherto (in socialist countries this demand is already being met).

3.8 Role of the curricula

All these developments pose many problems relating to the construction of curricula and their continual revision. Almost everywhere one sees overloaded curricula, which are one of the major obstacles to the modernization of teaching methods, because methods based on the organization of the pupils' mathematical activity require a relaxed, unhurried atmosphere. There is a demand, therefore, for curricula outlines, flexible and open to the teachers' own invention. But, on the other hand, we must take account of the classroom reality and the qualifications of the mass of teachers in the middle-year classes, their compulsory timetables, and their own conditions of life. Many teachers follow the text-book page by page, which thus replaces the curriculum. And the text-books are often overflowing with material. It would be unjust to burden the teachers with a responsibility beyond their qualifications. Childrens' opportunities in their mathematical education should not only depend on the teachers' individual opinions of what is more or less important in this education. So intermediary solutions are being sought (hence the appearance of ideas such as "content-core", "rallying points", and "optional chapters"). The plan of the mathematics curriculum for the elementary school in Hungary also provids an original conception of this point of view (*Orszagos Pedagogical Intezet*, 1975).

4 METHODS, AUXILIARY RESOURCES, ASSESSMENT

In volume III of *New Trends in Mathematics Teaching*, there is an outline of the trends and problems concerning methods and teaching resources in the learning of mathematics. The outline being still essentially the same, there would be no point in repeating this excellent presentation here. Some recent observations are nevertheless worthy of mention.

4.1 Text-books

The avalanche of texts printed for the pupils' use, text-books, work-cards, programmed texts, associated texts intended for teachers, etc., creates problems for pedagogic research. There is a need for criteria to assess the text-book intended for the pupil, in general and specific aspects, as a mathematics book used as a means of learning. The mathematics text-book at the middle-year class level should fulfil a role, not only in the mathematical education of the child but also in his general education. The primary school child can do without a "pre-scientific" book; the student at the age of 17 or 18 should already be able to read and use a book during his studies. The pupil in a class in the middle-years should therefore be initiated into such techniques. The school text-book which presents an account of some part of knowledge is an important facility for "learning to learn".

But the text-books available are not very often written for the pupil, from this point of view. They are aimed rather at the teacher, in spite of appearances. Moreover, we have still no clear awareness of the working of mathematics text-books and of the criteria for their assessment.

The new text-books look spectacular with their colours, their ingenious drawings, their pictorial form, etc. But there is no analysis of how these aspects work, and of their role in the learning situation. The same remarks concern other texts and other auxiliary resources.

The first wave of the reform enriched in a revolutionary manner the written language of school mathematics by pictorial means (various diagrams, graphs, tree-structures), useful and appropriate to the level which interests us here. Some criticisms are to be noted, nevertheless, for these means of expression, used in exaggerated and incorrect ways, can give rise to serious misconceptions.

4.2 Streamed or mixed classes

The democratization of post-elementary teaching raises the problem: streamed or mixed classes, differentiation or equality, collective or individual work? In some countries external differentiation, and hence the division of pupils into parallel classes streamed according to their abilities, is institutionalized by the structure of the school. As a rule, differentiation of a certain type (for example, in the Federal Republic of Germany) should ensure the "permeability" of the system, that is provide the child with the possibility of being able to move from one class to another parallel one as he progresses in the learning situation. Such differentiation must be distinguished from the practice widely in force, for example in the United States, namely, the organization of parallel classes following varying curricula. The pupils can choose the more advanced classes or the more ordinary, or can take an accelerated course in some subjects which are taught in other classes over a longer period.

Optional external differentiation also takes place in systems which are generally unitary (for example, "mathematics" classes for pupils whose mathematical capabilities are revealed very early within mixed classes, or optional classes outside the compulsory learning situation for all pupils in the U.S.S.R. and in some socialist countries, etc.).

External differentiation is to be contrasted with internal differentiation in a mixed class. Many models of such differentiation are being studied and used: temporarily streamed groups, individual learning based on work-cards concerning the same basic subjects but differentiated according to their presentation, choice of exercises, optional subjects, etc. Notable in this field is the system of graded ability work-cards used in Scandinavia, ensuring a "permeability" of the different courses.

There exists also an opposition in principle to institutional differentiation, independently of its external or internal forms: according to the authors of the Japanese report, each planned differentiation leads to discrimination of some pupils (Japanese Report). It is stated that the "permeability" of the system of any external differentiation is only a pedagogic illusion. The child of 12 years, put into one of the "weak" classes, has only minimal opportunities of moving into a parallel class at a higher level for the whole of his school life.

Some recent work aims at harmonizing the indubitable values of learning in a mixed group with the need to ensure that all possibilities are left open to the individual mathematical development of each child. Along the same lines, the method recommended is that of mathematical situations sufficiently rich to allow each pupil to work at a level within his grasp in collaboration with the others. But such a natural differentiation, often based on the spontaneous division of the collective work, and not institutionalized, is difficult for the teacher and demands pedagogic invention.

All these attempts reflect problems very deeply rooted in the social foundations of the school "for all".

Theoretically, differentiation "at its limits" is only the individualization of the learning situation. This idea, recommended by certain experiments during the last ten years, has been seen to decline somewhat. Moreover, it seems that the integration of very differentiated teaching methods is the optimum solution for the level which interests us here.

4.3 Free mathematical activities

The application of the various forms of organization of mathematical activities outside class or even school for interested pupils has been in some countries a particularly important element of natural, healthy differentiation in mathematics education. Mention must here be made of mathematical clubs and reviews for pupils, the Olympiads at various levels, etc. The uniformity of school learning in some countries (the compulsory curriculum, a choice of only two or three text-books for the teacher) is complemented by a variety of other courses in mathematics education open to all who wish to take advantage of it.

4.4 Collections of problems

We must mention certain progress made in the construction of exercises and problems adapted to modernized curricula (for example, *IREM de Strasbourg*, 1972–1976). The reforms of the years 1960–1970 have eliminated many problems from so-called "traditional" mathematics which could not be integrated in the modern content. They have been replaced by the often sterile exercises of pure terminology. That is why the "problem of problems" today concerns, among others, the construction of problems available to average pupils, and devised not only with a view to consolidating the skills and exercising the abilities, but directed towards the development of attitudes and heuristic procedures, processes such as generalization and specification, the discovery of analogies, and the stimulation of various forms of reasoning. There is also the question of collections of open problems, simple but instructive applications, etc. In spite of some progress made in this field the current situation still demands a lot of work. International exchange in this would be extremely useful.

4.5 Assessment

An objective assessment of the results of learning at the post-elementary level, at the transition stage from childhood to adolescence, starting from general studies and going on to studies directed towards the future profession, presents perhaps even more difficult problems than those of similar assessments at other levels. A certain lack of confidence is observed in diagnostic as well as prognostic values of mathematical tests, if these tests are not complemented by other means of evaluation. The criticism of standardized tests, expressed in the NACOME report (NACOME, 1975), and in the opinions of some American teachers (United States Report), exemplifies this very characteristic point of view. But tests are applied in many countries, often with other means of assessment, when it concerns the pupil moving from one class to the following one or from one teaching stage to a higher one.

An endeavour to limit the quantity and the role of examinations is being gradually applied, parallel to the increasing role of the teacher's judgement. A continuing assessment is recommended based on observation of the pupil during his mathematical activity, stimulation of auto-control and auto-assessment, consideration of the collective opinion of the class, etc. What happens in the class-room is still far removed from the realization of such proposals. Assessment is still too much directed towards selection, which often leads to social selection. Frustrations are noticed in the pupils, dread of mathematics, discouragement in those who, being "good pupils" at primary school, become "bad", and traumatized by poor results at secondary school. The problem cannot, however, be solved by the demagogic command "down with assessment", for in the current structure of societies assessment of abilities is necessary. We must nevertheless minimize the harmful aspects of such an assessment and replace selection by pedagogic means by practices aimed at giving good guidance to the pupil with regard for his potential and his interests.

5 EDUCATION OF TEACHERS

It is impossible to sketch out the current very diverse situation concerning the initial training of mathematics teachers for the level we are considering in this chapter. It is often an intermediary education between that of primary school teachers and that of teachers for the higher stage of secondary school. It is provided by different institutions, universities, teacher training colleges and pedagogic institutes. This intermediary position entails certain consequences. The elementary school teacher receives mostly an intensive pedagogic preparation at the expense of his mathematical instruction. The scientific qualifications of the teacher at the higher stage in the secondary school are important, but generally his pedagogic education is tenuous. The mathematics teacher in the middle-year classes often pursues studies which are unsatisfactory from both the point of view of his mathematical training and his pedagogic training. And the level of his teaching demands precisely a multilateral preparation, given the transitory character of mathematics education at this level and the large numbers of very different children who attend the middle-year classes. Obviously initial training is only a stage in the permanent training of the teacher; but it is a very important stage and deserves thorough analysis.

Let us here emphasize the progress made in permanent training and in the help given to teachers in service. Retraining courses, lectures, meetings, wide use of television, correspondence courses, evening classes, numerous publications; and these are only particular examples of the different forms used to this end. Never before did the teacher have such possibilities available to improve or complete his initial training. But perhaps he has never before been so burdened by his professional work, and never have society's demands on the school been so great. What is therefore offered to teachers in service is not exploited to a sufficient degree. Without changing the social and economic status of teachers of children in the middle years, the mathematical education of these large numbers of pupils will not be improved.

6 RESEARCH PERSPECTIVES

During the preceding considerations many problems ready for research have been revealed. Above all, we feel the need of a thorough knowledge of the process of mathematics learning as it takes place at the levels treated in this chapter. There are enormous gaps in our knowledge of the pupils' difficulties in mathematics, of the sources of methodological misconceptions, of the natural thought which should be the starting point towards the formation of mathematical thought in the pupil. All this concerns a reality which is very difficult to penetrate, since it is particularly linked to the experience of the individual. But it is these very gaps which often force us to grope blindly through the reforms, with the aim of improving the mathematical education of the masses of adolescents. Research into the process of mathematical learning in the transition period from the thinking of the child to adolescent thought seems particularly important today, now that the secondary school is open in many countries to all normal children.

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PANEL

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Chapter III

Mathematics education at upper secondary school, college and university transition

D. A. Quadling

1 INTRODUCTION

Previous chapters have emphasized the development of mass education in primary and the early years of secondary schooling, and have considered the role of mathematics in the curriculum of all pupils, whatever their ability, social background or ultimate career. At the next stage, with which this chapter is concerned, a new situation arises in which there is a marked differentiation between the courses which different groups of students choose to follow.

In many countries the beginning of this stage coincides with the age limit of compulsory school attendance, so that the students have made a conscious choice to continue in full-time or part-time education rather than to take up full-time employment. Even where students are required to continue full-time education, they often have considerable freedom in the choice of the subjects which they will study, and may if they wish omit mathematics from their curriculum.

The institutional framework within which courses at this level are offered varies widely. In some countries, students remain in the school in which the earlier part of their secondary education took place, but in separate classes; others designate special colleges for work at this level. In countries where for a variety of reasons (for example, the need for pupils to study in a language other than the mother-tongue) progress towards the end of general education takes place more slowly, this stage of education may be associated with the first year or two of university studies. The issues to be raised in this chapter are, however, to a large extent independent of the institutional context.

It will be helpful to identify four types of course at this level from the point of view of mathematical education:

(a) *Academic courses,* which are usually planned to meet the needs of students who will subsequently proceed to university studies. Such courses tend to present mathematics as a specific isolated discipline, and to emphasize theoretical principles and the logical coherence of the subject.

(b) *General courses*, followed by pupils of not more than average ability but with no particular career objective. These courses are assuming greater importance as more pupils choose to continue their education beyond the age of compulsory school attendance. They aim to emphasize aspects of mathematics of special relevance to the ordinary citizen in the modern world.

(c) *Technical courses*, designed to provide the mathematical knowledge needed in particular areas of employment. Some of these courses (for example, those for engineers, business or the electronics industry) may have a substantial mathematical content, but the emphasis will be on application of mathematical processes to particular situations. Successful students may go on to pursue these studies to a higher level in a technological university.

(d) *Skill courses*, normally taken on a part-time basis by employees following particular crafts, and providing the basic mathematical skills needed (calculations, measurements, application of formulae). The mathematics will not usually be more advanced than the student has already met at school, but it will be related to one specific area of application.

It should be noted that there is not necessarily a one-one relationship between type of course and type of institution.

It is difficult to make a distinction between "academic" and "technical" without implying a value-judgement, and in some countries the view is taken that the existence of an élitist academic programme may hamper the implementation of much-needed technical developments. There is in consequence a tendency for the dichotomy to become less sharp at this level of study.

2 GOALS

In most countries, it is the practice to set down specific goals for mathematical education at various levels (1). Typical are those recently introduced by the Austrian Ministry of Education, which refer to

> "precise and critical thought, logical deduction, precise verbal expression, independent intellectual activity, creativity and intuition, insight into mathematical subject-matter and method, knowledge and abilities required for other subjects and for various real-world applications"

as goals appropriate to academic courses, and specify amongst the implications of these

> "knowledge of the axiomatic method, training in proof, training in the capacity for abstraction, familiarity with language and symbolic expression in the subject, fluency with the most important manipulative procedures, mathematical treatment of problems from natural science, technology, economics and other areas, development of geometrical intuition". (Trans.)

Few would question the desirability of these goals. Other qualities which have been mentioned in a similar context include general attributes such as "objectivity and self-criticism, perseverance, philosophical depth, mental agility, problem-solving skills, ability to learn from books, development of the collective spirit", and others more specific to mathematics such as "appreciation of mathematics as a basis of civilization, development of judgement in formulating and interpreting mathematical models, providing a feeling for the power of mathematics, encouragement of good attitudes towards mathematics". These goals reflect the status of mathematics in the view of society at large. From France, for example, we may quote from the preliminary report of a ministerial commission (2):

> "Mathematics enjoys a privileged role for our understanding of reality . . . . By its very nature it involves an economy of thought which permits us to classify, command and synthesise, often in a few short formulae, a body of knowledge which would otherwise resemble an encyclopaedic dictionary of infinite mass." (Trans.)

In similar vein, the standing conference of Minsters of Education for the Federal Republic of Germany reported, in a document (3) giving recommendations and guidance for the modernization of mathematical education (1968):

> "Only when an individual has, from an early age, gained insight into the thought-patterns of natural science and an appreciation of mathematical structure can he solve the problems with which he will be faced in the modern, rationalized world." (Trans.)

It seems important to set alongside these statements, however, the growing recognition that, at the stage of education with which this chapter is concerned, few of the goals detailed above are likely to be achieved in any measurable degree by more than a small proportion of students. Because of the prestige accorded to academic courses in many countries and their role as the gateway to university education, these courses are frequently followed by students who are far from fluent in the language and symbolism of abstract mathematics and for whom formal operational thinking is not confidently established. The optimism with which more formal, abstract mathematics was introduced in many countries a few years ago is giving way to a degree of scepticism amongst teachers and a growing awareness that the students who follow courses are not yet fully mature. Moreover, some countries report the shortage of mathematics teachers at this level as a further limitation on their ability to bring mathematical education up to the standard implied by the goals officially laid down – a shortage which is all too easily perpetuated by "exclusive" attitudes to the subject.

Thus whilst the identification of goals has an importance for curriculum developers, teacher trainers, writers of syllabuses and authors of textbooks, it is recognized that many students and their teachers work to more limited goals: namely, to acquire the knowledge and skills set out in the syllabus and textbook and the ability to apply such knowledge and skills to certain specific situations. Aspects of mathematical training which it has not been found possible to incorporate into the textbook and exercises, and which are not tested in the students' examinations, are unlikely to feature prominently in mathematical education at this level. These points are taken up again in section 5 of this chapter.

There is also a conflict, which is confusing the discussion at the present time, between the goals stressed by mathematics educators and those valued by employers or teachers of other subjects. Professors of engineering or chemistry in the university often look to the upper secondary school to provide their students with the mathematical skills and processes which are used in their own disciplines, and show little interest in rigorous foundations or a structural overview of mathematics (even if these aspects might have importance for the application of mathematics in the long term).

This divergence of goals is leading to increasingly vocal criticism of mathematical education in certain quarters, and even to the establishment of short-term "remedial" solutions in some university departments. Underlying this crisis of confidence is a fundamental question: does "mathematics as an education" differ in essence from "mathematics as a qualification"? This is an issue on which there is a need for wide-ranging and open-minded debate amongst all interested parties.

As might be expected, there is a stronger concensus of view in technical courses, and indeed in some countries industry plays a large part in framing the goals and in the examination of students; it may also make a substantial financial contribution to their training. Goals for mathematics in technical education stress exact, logical and critical thought, the ability to apply mathematics to problems in the particular field, and reliability in using numerical, algebraic and graphical techniques. In many countries there has been little change in recent years in the goals of mathematics in technical education, though its practice has been considerably influenced by the growing importance of the computer.

3 MATHEMATICS CURRICULUM AND CONTENT

3.1 Mathematics within the total curriculum

The structure of academic education at this level varies considerably from country to country, but everywhere mathematics is regarded as a fundamental component of the curriculum. Sometimes (as in Denmark, Austria and Brazil) mathematics is compulsory below the level of university; typically 4 or 5 hours a week are set aside for the subject. It is, however, in many cases possible for students to offer it to different degrees of intensity: for example, in the Federal Republic of Germany, mathematics may be taken either as a principal subject in the *Abitur,* as a third- or fourth-choice subject, or at a subsidiary level involving only three semesters in the three years of the course in some *Länder;* in the French *Baccalauréat,* different amounts of mathematics are taken in the sections for natural science (7 hours per week), economic and social sciences (5 hours) or philosophy and letters (3 hours); and in Czechoslovakia, the course in the science branch involves more mathematics than in the humanities branch. In other countries, only students following certain particular sections of the academic programme are required to take mathematics. In the University of Lesotho, for example, all science and economics students have to offer one year of mathematics, and physics students must offer two years; a rather similar pattern is found in senior secondary schools in Sri Lanka. In countries where students have apparent freedom to design their own curriculum, their choice is often in practice restricted by considerations of university entrance. Thus in the United States, students with college ambitions are advised to take mathematics through to 11th grade, and those whose interests are oriented towards science are advised to continue with mathematics right to the end of high school; otherwise they may be required to pick up in college mathematics qualifications they have missed at school. Many United Kingdom universities will only accept for physics and engineering courses students with a good pass in the final school examination in mathematics, whilst for economics, geography or biology preference may be given to applicants who have studied mathematics to the end of their school education.

In describing the situation in various countries, it is important to recognize internal forces which help to shape curricular policy. One factor is a shortage of teachers which may restrict the amount or the kind of mathematics that can be offered. Students' own aspirations also have an increasing influence. A society whose prosperity depends on the widespread acquisition of high mathematical skills cannot recruit its students solely from those who have a strong leaning towards mathematics; it is therefore important to relate the teaching of the subject to the students' total curriculum and to take their wider concerns into consideration.

The effect of recent changes appears to be a trend towards a mean position. Countries such as England which have had a tradition of highly specialized studies at this level, with mathematics taken by relatively few students, are moving towards broader curricula in which mathematics can be offered in association with a wider spread of subjects. On the other hand, in certain countries where it has been the custom to prescribe a common curriculum for all students, more choice is now being given as to the depth to which mathematics is studied, or even whether it is studied at all.

3.2 Variety of mathematics courses

There is a particular problem in planning the mathematics to be taught in technical courses, since students have already made a decision to take up a particular kind of work, and are therefore more likely to be well motivated if they can see their mathematics in that context, especially if the teacher is also knowledgeable in the area of application. However, there is much common ground between the mathematics required for the various applications, both in content and in the emphasis on models and algorithms. It is therefore almost inevitable that, on grounds of organizational economy, students with different specialisms will learn much of their mathematics in the same class, using a general textbook rather than one directed to their special interest. It may happen, however, that there will be some separation into specialist classes in the later years of the course.

Although some countries still have the same mathematics syllabus for all students following academic courses, there is a general trend to allow a degree of choice between different branches of mathematics in the senior high school. A common arrangement is to specify a core of material which all students are required to study – some calculus, algebra and trigonometry, perhaps – which might take up about one-half of the course, and then to allow choice between a number of options such as statistics, linear algebra, abstract algebra, numerical analysis or informatics. The choice may in some cases be determined by the mathematical requirements of other subjects in the students' curriculum. However, it has been pointed out that many senior high schools are not large enough for a wide range of options to be made available; in this event, the choice is in practice made by the teacher rather than the students. Moreover, if there is too much variety in the work done at this level, problems arise at the next stage of education; university teachers sometimes complain that there is little common knowledge amongst the students in their first-year classes.

An interesting example of this kind of course structure is provided by the International Baccalaureate, an examination devised originally for use in schools with students of many different nationalities (4). The curriculum for this examination comprises six subjects, three at "higher level" and three at "subsidiary level". All students must offer mathematics at one or other of these levels, but at subsidiary level this may take one of two forms. The first of these is along more or less conventional lines; this and the higher level course have a "core plus options" structure, and it is of interest that in the current revision of the syllabuses the amount of core material is being substantially increased, and the options correspondingly reduced. The second subsidiary level, however, intended for students who do not expect to make use of mathematics in their subsequent careers, is being re-designed under the title Mathematical Studies. This will emphasize the cultural roles of mathematics, and will permit students to concentrate for a substantial part of their time on a personal project covering an aspect of mathematics or mathematical application of their own choosing.

In some countries a general course in mathematics is also available, usually intended for students at a lower level of mathematical ability than the academic course and lasting for a shorter time. These courses tend to place more emphasis on the applicability of mathematics rather than on its logical structure. There is often a possibility for students to choose a particular area of mathematical application, such as business mathematics, technical mathematics, statistical methods, mathematics in design, consumer mathematics, etc.

3.3 Provision for mathematically gifted students

A special problem is that of providing (either within or outside the school) means of stimulating and deepening the mathematical experience of those relatively few students who display particular aptitude for the subject (5). It is often remarked that mathematics is a "young person's subject", and history records many examples of precocious young mathematicians; one is therefore concerned that they should enjoy a challenging mathematical diet which will enable them to build up their strength. Some ways in which this is being done are listed here.

(a) A few countries make extra time available for mathematics for selected students within the curriculum or in small groups outside regular school hours. This may be provided either in special schools (as in the U.S.S.R.) or as a special section in some ordinary schools (as in England) or both (as in Czechoslovakia). A good feature of these arrangements is the opportunity for students to work regularly with more than one teacher. However, it is not always easy to find teachers qualified to take these classes, and some countries have reported declining support amongst students, teachers and administrators for this form of special provision.

(b) Where there is sufficient flexibility in the curriculum, as is often the case in United States schools, good students may be able to take mathematics courses a grade early, making it possible for them to take "advanced placement" courses in their senior year which for most students would be deferred until the university. Alternatively, short "enrichment" courses may be injected into the regular curriculum for selected students. In large schools, too, some ability grouping may be practised so that some classes can study the subject in greater depth than others.

(c) There has been a rapid spread in recent years of Olympiads and other mathematical contests (6). In 1975, teams from 17 countries took part in the International Mathematical Olympiad, but this is only the top of the pyramid; in each country, the selection of the international participants is made through a series of national tests often affecting thousands of students.

(d) It is not uncommon for distinguished mathematicians and oustanding teachers to lecture to groups of students, perhaps assembled together from groups of schools for the purpose. This may occur regularly as part of the curriculum, or at a special occasion organized by a university or a local mathematical society, in the form either of a short conference or a single lecture. An extension of this idea is the summer school, in which a more systematic programme of study can be followed. The value of opportunities for students to work alongside their peers cannot be over-rated.

(e) Mathematical journals written specially for students, and often containing contributions written by students, are another means of stimulating interest in mathematics (7). These vary from established journals with a national distribution to more ephemeral publications produced within a single school or university.

(f) Some national organizations of mathematics teachers provide support (e.g. by circulating suggestions for problems) for school mathematics clubs.

It is clear that there is a growing awareness of the need to interest able young people at high school in mathematics as a subject, and also to encourage able students to take an interest in its applications, particularly in new fields such as biology, environmental science, statistics and operational research.

3.4 Style of the mathematics course

There are three aspects of mathematical didactics at this level which have been the subject of wide debate in recent years. These are the degree of abstraction appropriate to the course, the possibility of an integrated presentation (within mathematics) of the subject-matter and the place of deduction and axiomatics.

3.4.1 *Abstraction*

It is a truism that mathematics progresses by taking an increasingly abstract stance, and this has never been demonstrated more clearly than during the present century. Abstraction provides a "short cut" to the frontiers of knowledge, by furnishing the means to incorporate within a single theory a number of special cases and by focusing attention on those features which seem to be essential and suppressing those which are superficial. Moreover, it is the abstract character of mathematics that makes possible its application to problems in a wide variety of fields outside mathematics. But it is precisely here that the educational problems arise, for such applications bring with them a demand for a high level of mathematical literacy amongst an increasingly large proportion of students; and for many of these students abstract mathematics is uncongenial and perhaps even inaccessible. The saving of time, which is one of the reasons given for adopting such an approach, results in a course which for many students is pure book-learning and one-way instruction, so that they never successfully acquire the intended concepts. Indeed, if the goal of more effective problem-solving is never achieved, it may be questioned what has been gained by demonstrating the abstract concepts.

There are many teachers who would assert that, whilst the ability to handle mathematics to a high degree of abstraction is a desirable goal of mathematical education, abstraction is more properly described as an activity than as a state. For such teachers, it is more important to involve students in the *process* of abstraction than to teach them about the abstract structures of mathematics; and this would require learning materials of a very different kind from most text-books in current use at this level.

Certainly it has been a common experience in many countries that some of the original enthusiasm, shown in the early days of curriculum reform around 1960, for courses in the senior high school on topics such as function theory, vector spaces, groups and fields has declined. It is more widely recognized that the step from the mass education of the lower secondary school to a level of education in which there is some element of self-selection does not in itself turn students into fully-fashioned adult mathematicians, able to operate at an abstract level without extrinsic motivation, meaningful illustration and practice in the development of supporting skills. Instead, the students are seen to be in an important phase of development, in which formal operational reasoning still needs to be consolidated and the ability to handle language developed to the point where it is adequate for the sophisticated purposes of higher mathematics.

In this context, it has been remarked that the qualities required to succeed in such mathematics courses – mastery of language and of oral expression, quickness of assimilation and familiarity with abstract ideas – are those more likely to be cultivated at higher levels in the socio-economic scale. Since the prestige of mathematics gives it a major role in selection for many of the leading professions, the premature introduction of an abstract approach into mathematical education can have a conservative effect on the broad structure of society.

As a reaction to the highly abstract courses of recent years, it is becoming more common to find syllabuses for the senior high school in which there is a more concrete, process-oriented programme of mathematics until the final year, when an attempt is made to summarize and show the connections in what has been learnt, from a more abstract point of view (8). (It has to be admitted, however, that under pressure of time this last stage may get little attention in practice, especially if the examination at the end of the course puts an emphasis on the ability to perform routine calculations.)

3.4.2 *Integrated approaches within mathematics*

Practice varies from country to country between syllabuses divided into a number of sections, each concerned with one particular branch of mathematics, and the presentation of a syllabus simply entitled "Mathematics", in which the material is organized on an implicit sub-structure of abstract concepts. For example, in syllabuses of this latter type, students may meet in close succession ideas of linear dependence of equations, motion of a particle in two dimensions, some theorems of affine geometry and the equation of a line in 3-space; clearly an abstract view of the vector concept underlines this choice of topics, although the student may not be made consciously aware of the properties of a vector space at this stage. To illustrate this difference in approach we may contrast the subdivision of the syllabus for the *Kollegstufe* in Bavaria (9) into sections separately entitled Analytical Geometry, Boolean Algebra, Probability, Statistics, Infinitesimal Calculus, with the fact that there has been no textbook published for English sixth forms for many years specifically on "Calculus", "Algebra" or "Analytical Geometry", but only on "Mathematics" or "Pure Mathematics" (since in some schools the applied subjects Mechanics and Statistics are still taken separately).

The division into separate branches has obviously certain practical advantages: in particular, it is easier in this way to offer students some choice in their mathematics programme, and it reduces the problems which students experience if they have to move from one institution to another. Whichever policy is adopted, however, there is an increasing trend towards emphasizing the unity of mathematics. For example, in laying down the didactic principles which underlie the new syllabus for academic courses in Austrian high schools, we have the statement: "The student should understand clearly the unity and the mutual relationship between the individual branches of mathematics, and be able to draw structural connections from this." (Trans.) This is typical of the policy in many countries.

Some teachers, however, take the view that an appreciation of the unity of mathematics in conceptual terms is an unattainable objective for students who are still mathematically immature. They prefer to demonstrate this unity in applications and problem-solving, where the student can see how different branches of mathematics have different uses and yet relate to each other.

3.4.3 *The place of deduction and axiomatics*

This is an aspect of teaching in which national traditions tend to persist through changes in syllabus, and there are wide differences in practice. Questions such as "which definition of continuity should one use?" or "what is the best way to formulate axioms for affine geometry?" may be warmly debated in some countries but appear quite irrelevant in others.

There is some indication that intuitive approaches are being used more widely in the introductory stages of subjects such as calculus and probability, formalization being deferred until students have some experience of applying the concepts and have become familiar with the appropriate language. Axioms and definitions can then be built up gradually as they are needed, the students themselves participating in the process.

The role of axioms at this level is dependent in some measure on the way in which the syllabus is organized (see section 3.4.2). With courses divided into separate branches it is possible, if one wishes, to introduce early on a statement of definitions and axioms (or, less formally, "properties") on which subsequent work is to be based. In integrated courses, on the other hand, a structure imposed by deductions from axioms is very likely to conflict with that suggested by clustering topics round concepts; the instances of deductive mathematics in such a programme are likely therefore to be on a "local" rather than a "global" basis.

3.5 Topics covered in the course

There is considerably less difference between countries in the content of the syllabus at this level than there is in school organization or the style of presentation of the material. We have now reached the situation where, in those countries in which syllabuses are issued by the Ministry of Education, nationally or by states, a new syllabus has been in force for a few years (usually the result of controlled experimentation in a small number of schools) following the widespread discussions about the "new mathematics" in the period 1960–1965. There has thus been time to observe the effect of the change in syllabus when implemented across the whole school system, and some countries have already taken steps to revise the new syllabus in the light of this experience. In countries where individual schools, or local school districts, have autonomy in determining syllabuses, a substantial proportion of schools have gone over to a new programme, in whole or in part. One can therefore hope to review the situation with a reasonable measure of objectivity.

Before describing the main changes in the new syllabuses, it should be stressed that there is always a body of basic mathematics, which remains relatively fixed, around which these changes have been made. In many countries, this covers differential and integral calculus, analytical geometry, polynomial theory and combinatorics. Associated with such a basis it is now common to find:

Probability	(theory for discrete sample spaces, possibly including Markov chains; binomial, normal and Poisson probability models)
Statistics	(descriptive statistics, statistical measures, elementary tests of significance, point estimation and confidence intervals)
Linear algebra	(sets of equations up to 3-by-3, matrix notation)
Vector space theory	(including the notions of basis and dimension, but without full proofs)
Algebraic structures	(groups, rings, fields)

and less commonly:

Boolean algebra	(with emphasis on applications, propositional calculus)
Optimization	(simple linear programming)
Informatics	(with emphasis on scientific applications of computing)
Affine geometry	
Numerical analysis	

In addition to the introduction of new subject-matter, there have also been many changes in the ways in which various topics in the old syllabus are treated. These are dealt with in more detail in section 4.1 below.

To make space for this new material, the subjects most commonly dropped have been solid and spherical geometry, conic sections, commercial applications, and trigonometry associated with the solution of triangles and with the manipulation of trigonometrical formulae. It is clear, however, that in many cases the newly introduced material is more demanding, both in quantity and conceptually, than that which it replaced. A consequence has been a reduction in the performance of exercises and in problem-solving – either intentionally or merely through lack of time. Another matter for concern is that this change in subject-matter may reduce students' opportunities for using geometrical language.

In the most recent revisions of syllabuses some new material, such as elementary theory of sets and functions, which was introduced when new syllabuses were first devised, was moved to an earlier stage and is now taught in the lower part of the secondary school. There has also been a trend to discard from this new material some of the more abstract subjects such as algebraic structures and the theory of vector spaces, since these were found not to have been successful with the full range of students following an academic course in mathematics. Mention should be made at this point of difficulties which have often been experienced in teaching this new material by teachers who themselves lacked the necessary background of knowledge in these new subject areas; whilst syllabus revision can have a beneficial effect on teachers and provide a strong incentive to take further in-service training, the difficulties which it poses for teachers are an important practical constraint on the speed with which syllabus changes can be promoted.

Syllabus changes in technical courses have come more gradually, and have been far less drastic than those described above. Since the courses sometimes vary extensively even within a single country, it is difficult to say much about this in a general survey. There has, however, been some attempt to introduce ideas of set and function into the courses, with the aim of helping the students to a better understanding of conventional arithmetic, algebra and calculus; and subjects such as statistics, probability, Boolean algebra, vectors and informatics have also sometimes found a place. In some countries the availability of computers has had an important effect on the mathematics taught in technical courses, making it possible to give a numerical rather than analytical emphasis.

3.6 The balance of the mathematics curriculum

It has already been mentioned that, in revising the new syllabuses, some adjustments have been made away from the more abstract forms of mathematics. Reference has also been made to the reduction in the time available for practising exercises and problem-solving, which has brought criticism from some university mathematics teachers and teachers of other subjects (especially the physical sciences) that students are not sufficiently fluent in important manipulative skills.

We are faced here with a conflict of interests. It is suggested that, to be widely accepted, the "academic" mathematics curriculum must strike a balance between the increasing abstraction and logical precision of modern pure mathematics, the process skills and knowledge of specific applications required by traditional users of mathematics in engineering and physical science, and the more open interest in the general applicability of mathematics shown by newer users in the biological, social and environmental sciences.

However, although in some countries there has been strong involvement of university mathematicians and of industrialists in experimental curriculum developments, it is remarkable how narrow a range of interests is often represented on national committees which take decisions about the mathematics curriculum. In some countries universities exercise an influence, either through official representation or through their right to set tests to decide who may enter the university. But it is a matter of concern that there are so few instances where positive steps are taken to try to ensure that the mathematics taught in school is in tune with the needs of society at large, or even that it is a good preparation for those students who wish to use their mathematics as a support for work in other fields.

In this context, the longitudinal studies being conducted for the Caribbean Examinations Council into the performance of candidates in the initial years of employment or further training and its invitation to civil service, business and manufacturing associations to comment on syllabuses, are developments of special interest. Mention may also be made of an enquiry in Ontario, Canada, comparing the characteristics of high school mathematics with the expectations of university teachers.

3.7 Integration of mathematics with other subjects

Would it be possible, for those students who study mathematics mainly for the sake of its usefulness in other subjects, that the mathematics might be absorbed into the teaching of the other subject?

It appears that few countries have experience of such an approach, though there is occasionally a precedent in vocational courses strongly linked to a particular field of application, such as electronics or applied biology. In such cases, however, the mathematics is usually limited to techniques needed for a particular purpose, such as complex numbers or some statistical tests, rather than constituting a complete systematic course.

There are major difficulties in an integrated approach in academic courses. Not the least of these is the long tradition of the separation of disciplines in many countries, so that teachers at this level have often had a specialized preparation in only one subject and are wary of committing themselves to interdisciplinary co-operation. Moreover, the belief is strongly held that any subject studied at this level should have a sound theoretical development, and there is a fear that the frequent intrusion of ideas foreign to the subject may result in confusion. Another danger is of "mathematical imperialism", trying to find mathematics everywhere and obscuring concepts natural to the other discipline by the premature imposition of mathematical structures upon them.

Nevertheless, it is felt in some quarters that the motivational advantages could be considerable. In France, for example, there have been discussions with teachers of philosophy, language, economics, geography and the various sciences towards the design of an interdisciplinary education, and the facilities of some of the IREMs *(Instituts de Recherche sur l'Enseignement des Mathématiques)* have been used to initiate local experiments (10). In the United States, a project in biomedical education is developing a programme which incorporates the relevant mathematics within it. Another approach, developed by the Continuing Mathematics Project (11) in the United Kingdom, is to prepare modules of mathematics for individual study (programmed texts, tape-slide presentations, etc.) which can be used flexibly by teachers and students as part of a course in some other subject – either in sequence to build up a body of knowledge in subjects such as statistics or calculus, or individually when needed to provide the mathematics appropriate to a particular topic. It should be stressed, however, that these are experiments which have so far affected relatively few students.

In the long term, such developments require fundamental changes of attitude, not just on the part of mathematics teachers but within the educational world at large. It is not sufficient to regard mathematics as "serving" physics; the problem is to devise a structured curriculum in which mathematics, physics, biology, etc. combine to advance knowledge. All that can be reported so far is a growing interest in several countries in co-operative courses on these lines, and a small amount of experiment designed to find ways of overcoming administrative and educational difficulties. These issues are discussed further in Chapter XII.

4 TEACHING METHODS

4.1 Treatment of the course content

It is impossible to describe in detail the methods – taken in mathematical sense – used to teach the new syllabuses. Apart from the textbooks, these methods are extensively debated in journals for mathematics teachers. One can, however, identify certain common trends, and also observe certain contrasts between the practice in various countries.

One important factor in many countries has undoubtedly been the increasing availability of computers for schools and colleges. Apart from the introduction of informatics into the curriculum, much thought has gone into the possibilities which computers offer to devise new approaches to the teaching of certain mathematical topics. For example, a numerical search for the solution of an equation or the position of an extreme value, followed by an examination of the results (printed out, or shown on a visual display unit), may lead to the concepts of continuity and derivative in a manner which is more real to students than abstract definitions. The simulation of stochastic situations with a random number generator can bring a sense of reality to probability distributions by demonstrating both random variations and the trend towards stability. The wish to compute certain functions can be used to motivate the idea of a recursive definition. Tables of functions can be computed *ab initio*. (Many other possibilities are outlined in Chapter XIII.) In general, emphasis can be placed on the structure of a calculation whilst results are still displayed in numerical form. In North America, in particular, complete courses in subjects such as algebra, calculus and statistics have been constructed round the computer. It also has an important potential role in applications-based courses.

Some of the strengths of the computer in the ability to perform numerical computations are also possessed in lesser degree by pocket calculators, and these have the advantage of cheapness so that there can be several available in the same class – perhaps even one for each student. Groups of teachers have begun to explore the possibilities of using these calculators in mathematics teaching, though it is yet too early to draw any conclusions. There will still, however, be many occasions involving repetitive computations or random processes, or when the stress is on the organization of calculation, where a pocket calculator cannot be any substitute for a computer.

On a more theoretical level it is now common practice to place emphasis on the concept of function as a mapping from one set to another, using this as the means of introduction to the differential calculus and to trigonometry, and linking it also with transformations of the plane and a more dynamic approach to affine and Euclidean geometry. Examples with small finite sets are often used to give experience of the concept. In this approach, sine and cosine appear as functions, and are studied with an emphasis on their periodicity and symmetry, rather than as ratios of sides of a triangle.

Another important change is the extensive use of vector methods to lead in to the analytical geometry of the plane and of 3-space; the results in terms of co-ordinates and equations are then derived from the corresponding vector relationships. This in turn leads to a greater emphasis on affine properties.

A survey of number systems, exemplifying algebraic structures such as rings and fields, is now frequently included in the curriculum. Group structure is identified as a common feature in many parts of mathematics – number-systems, vectors, transformations, functions, etc. Complex numbers are often introduced through vectors or as transformations of the Euclidean plane, thereby forming another link between ideas of function and field structure; matrix notation may be used to illustrate this link.

An examination of textbooks reveals that the new syllabuses usually brought with them a greatly increased emphasis on precision of mathematical language in definitions and argument, sometimes to such an extent that students were repelled by the course and mathematics was in danger of becoming overwhelmed in verbalism or symbolism. In practice there has been a retreat to a more informal level of presentation, especially in countries where the exercises which students are expected to perform at the end of the course consist largely of direct applications or problems rather than the repetition of proofs of standard results.

Special reference should be made to a mathematics project initiated by the Schools Council for England and Wales, whose researches indicated that many students following the conventional course were being expected to cover new ground far more quickly than they were able to form the relevant concepts. This project has devised a new style for teaching mathematics, which has been called Applicable Mathematics (12). The emphasis is on hypothetical situations for which various mathematical models form an appropriate description, and many sections of the course in fact have titles such as "vector models", "linear models", "probability models", etc. The aim is to cover less ground than in the standard course, but to concentrate on building up concepts through a variety of real-life problems.

One of the major areas of debate is the choice of approach used in probability and statistics (13). The new syllabuses in many countries prescribe the definition of probability based on sets of elements in a sample space, from which a strong structure of probability theory is established before applications to statistical problems are discussed. There is, however, a school of thought which believes that it is undesirable for the teaching of statistics to be too strongly rooted in mathematics, and which would like to see the course growing out of the discussion of real data. Special mention may be made of the joint committee of the American Statistical Association and the National Council of Teachers of Mathematics in the United States which has sponsored a project on the development of statistical concepts through case studies (14). Much work remains to be done, however, in investigating how far such a programme can be taken, and how and when the conceptual development of statistical ideas should be given theoretical support (which may call on quite advanced mathematics). It is not easy – in the planning for students at a modest level of mathematical sophistication – to steer a middle course between the mere application of standard statistical techniques and a theoretical approach divorced from real-life problems.

To summarize this section, we should perhaps question to what extent the mathematical diet which is offered to students at this level contributes to an implementation of the goals referred to earlier (see section 2). Many of the changes described above appear to be motivated more by a desire to preserve the purity of mathematics than by the broader educational needs of the students. We should perhaps be concerned at the relatively small extent to which the new programmes appear to encourage "independent intellectual activity, creativity and intuition . . . objectivity and self-criticism . . . appreciation of mathematics as a basis of civilisation, development of judgement in formulating and interpreting mathematical models . . . encouragement of good attitudes towards mathematics" (see section 2). It may be more important, in the coming years, for those charged with the development of mathematical education to try harder "to see ourselves as others see us".

4.2 Teaching mathematics through exploratory methods

It is common experience in many countries that new syllabuses have made greater demands in terms of the amount of knowledge that students are expected to acquire; there is a danger that, whilst students may know more mathematics, they may be less adaptable in its application. It is hardly surprising therefore that whilst some interest has been shown in the possible role of exploratory investigation in open-ended situations in the teaching of mathematics at this level, little has been achieved. Nevertheless, groups of teachers in several countries have been studying possibilities of incorporating a component of "personal mathematics" into the course, taking the view that only in this way can some of the goals of mathematical education – self-reliance, confidence in using mathematics, the ability to use books and to work co-operatively in a group – be realized. Indeed, this method of working is now used extensively in primary education in some countries with the aim of helping children to develop enquiring attitudes in mathematics;

it is surprising that, at the stage where students are beginning to select themselves for their potential ultimately to follow a career involving mathematics, and have attained a capacity for formal operational reasoning, nothing further is done to foster research skills.

Work on these lines also points a way for mathematical studies to play a larger part in students' social education. In senior secondary schools in Sri Lanka, the application of knowledge acquired in the classroom to a project of service to the community is an integral part of the curriculum, and students learn through this that mathematics can be used to tackle real problems and to improve the quality of life of the population. Reports from other countries have corroborated the contribution that such work can make in terms of motivation, giving students a more positive feeling about their abilities, their role in society and the importance of learning.

One development which is undoubtedly helping to make such work a reality is the increasing availability of computers. If a student is practising computer programming, which is of its nature an individual and largely self-evaluating activity, then there is much to be said for allowing him to write programs to investigate problems chosen by himself rather than requiring him to perform exercises prescribed by the teacher. If these problems relate to real applications of mathematics, so much the better. Another area which lends itself particularly well to this style of work is statistics, where a student can collect his own data related to his personal interests and then select appropriate techniques for analysing and representing them. In England there are some final examinations which require students to present such personal work in computing or statistics as part of their total assessment, credit being given for the initiative shown and the form of presentation as well as for the correctness of the application.

The extension of such work to other areas of mathematics at this level has so far been largely undertaken by associations of mathematics teachers and by experimental curriculum development projects. The Association of Teachers of Mathematics in the United Kingdom has been notably active, and has now been invited to prepare a syllabus for the final two years of secondary school with a substantial component of exploratory work; it is hoped that this will not be seen as in competition with more formal elements of the course, but that it will provide positive support towards the acquisition of subject knowledge. The Applicable Mathematics Project, referred to above (see section 4.1) has proposed a set of procedures for marking such work in a more objective manner than was previously possible.

Typical investigatons carried out by students include mathematical aspects of various sports and games; patterns of numbers associated with a set of geometrical figures; the changes in shape of the graphs of a family as the parameter is varied; a variety of queueing problems; the layout of a car park; the mathematics of a hovercraft or helicopter; the effect on geometrical shapes of various transformations of the complex number plane; and so on.

There are clearly difficulties in establishing work along these lines as a part of the regular mathematics curriculum. It is very time-consuming, so that its adoption must involve some sacrifice in the formal teaching programme (though not necessarily a loss in the final competence of the student); and the qualities which are claimed to be developed in this way are not so easily measured as the acquisition of new knowledge or skills. Reliable evaluation of students' work is hard to achieve, especially if (as may be desirable) it is carried out in groups rather than by individuals. Above all, it requires a fundamental change in attitude to mathematical education on the part of teachers, many of whom have had no personal experience of "exploratory" mathematics and would see in it a threat to security in their relationships with students. Attempts have been made to overcome this in one or two countries by including such activity in teacher training, both pre-service and in-service, and by strengthening links between teachers and mathematicians working in industry. Nevertheless, this development must still be regarded as being in an experimental stage.

4.3 Individualized learning programmes

At the other extreme, there is growing interest in the construction of courses broken down into small stages, each with clearly stated behavioural objectives, and designed for each student to work through individually. Typically a student begins by working a pre-test to determine the skills which he already possesses, and is then directed to individually prescribed (sometimes individually chosen) learning materials which he studies until he successfully demonstrates mastery of the objective. Ideally these programmes offer variable pace of instruction and variable media of instruction adapted to the needs and aptitudes of each student.

Apart from their effectiveness as a teaching technique, such programmes help to promote self-reliance and to foster a spirit of co-operation between students. There are countries in which self-instructional programmes are specifically included in the senior high school course in order to promote these aims.

This approach to mathematical education naturally has its strongest appeal in technical courses, or where mathematics has a recognized "service" role in the students' curriculum. In these cases, the student is conscious that he needs to learn a specific piece of mathematics for a specific purpose, and is well motivated to follow a course explicitly designed to enable him to achieve that objective. It has yet to be demonstrated, however, that it is possible to devise programmes embodying this approach which can take the student beyond limited goals at the cognitive level and give him an overview of mathematics as a whole.

5 EVALUATION OF STUDENT ACHIEVEMENT

The course in an upper secondary school is in almost all countries completed by an assessment of the student's performance, in the form of a mark or grade, which may have considerable influence on his future career. Frequently this assessment is based solely on an examination, set and marked at a national level, in which the student is asked to apply his knowledge to the solution of a number of exercises without reference to books or notes. This examination may take the form either of a multiple-choice test or of a number of longer open-ended questions, or a mixture of both. The students may also be asked to reproduce proofs and definitions which they have been taught, but this is less common. In some countries, other forms of assessment are used, either instead of or supplementary to the examination: these may be based on tests and other assignments done by the student during the course and marked by the teacher, on an oral examination, or occasionally (but rarely) on projects carried out by the student in his own time.

The importance to students of the results of this assessment has the effect that most teachers and students put their main emphasis during the course on those attributes which contribute to it – usually the acquisition of knowledge and skills, and the ability to apply these in given situations. Other goals whose achievement is less easily evaluated by tests of this kind – the encouragement of creativity, perserverance, the ability to learn from books, the devising and interpretation of mathematical models – receive correspondingly less attention from teachers and students.

In some countries the grade awarded at the end of the course is used as the basis for deciding the fitness of a student for university education, but in others university teachers set separate tests for this purpose. This latter practice has a superficial attraction, since one would expect a prognostic test of aptitude for university studies to be different in kind from a test designed to discover the skills which a student has acquired in the upper secondary school. However, the practice can give rise to difficulties, since university teachers are not necessarily knowledgeable about developments in teaching at previous levels, and the style of a university examination may therefore distort the school curriculum. Moreover, the constant search for novelty in devising prognostic tests can easily lead to a concealed enlargement of the curriculum as schools respond to university demands. One may add that an enquiry in Scotland was unable to find evidence that specially designed aptitude tests, teacher assessment, or interviewing of candidates were more effective guides in selecting university students than the school leaving examination.

A not dissimilar problem arises in some countries in technical courses, where representatives of the industry take part in setting the examinations for the course. Whilst this would seem to be a wise procedure, lack of contact between teachers in schools and colleges and the industry can result in examinations which are not well suited to the course. Whatever the level or field of education, it is desirable that responsibility for the examinations and planning of the curriculum should be closely integrated.

It would seem that there has been little innovation in the techniques of examining mathematics at this level in recent years, except that increasing use is being made of multiple-choice testing — though usually in conjunction with other forms of examination, not as the sole criterion (15). It may be that the importance of the grade at the end of the upper secondary school course makes teachers nervous, and users suspicious, of experiment in examination procedures. This is a matter of particular concern since some of the newer material in the curriculum, such as statistics and computing, is less well suited to examination by conventional methods than the more traditional mathematics. There have been reports that the abolition of the external final examination has led to a decline in public confidence in standards, resulting in strong pressures for its restoration.

6 RESEARCH ISSUES

Whilst many of the new developments in mathematics at the primary level have roots in research into theories of learning and child development, it must be remarked that developments in the senior high school have concentrated almost exclusively on the subject-matter itself and its presentation. Changes in the syllabus reflect changes of emphasis in mathematics as practised by mathematicians, rather than changes in our understanding of students' thought-processes or motivation.

Nevertheless, work with experimental programmes in recent years has inevitably brought with it a greater awareness of students' reactions to a variety of mathematical learning experiences, and the adjustments made in these programmes are often the result of these observations. It appears to have been rare, however, for any record to have been made of these.

Attention has already been drawn in this chapter to the discrepancy between the declared goals and actual student achievement. There are questions here which should have priority in research effort. In the first instance there is need for an investigation, covering a wide range of interests, to identify goals for mathematical education which are realistic and relevant for the last quarter of the 20th century. Secondly, we need to understand the factors which prevent many students from achieving some existing goals.

One important distinction is between the learning of processes and specific skills and the development of cognitive strategies relevant to problem-solving. Research by Gagné (16) suggests that, whilst the former can be achieved through breaking the process down into small steps, the latter occurs through repeated involvement in problem situations. If valid, this conclusion clearly has implications for education at this level.

Another area of particular concern at this level is the conflict between the everyday language to which students are accustomed and the highly developed special language used to communicate mathematical ideas. At the level of formal operational reasoning the student is more than ever dependent on adequate linguistic support, and recent work in curriculum development has brought this problem to the surface. It is to be hoped that this will become a major area of research activity. Special problems arise in many countries in Africa and Asia, where the language of instruction in secondary school and university is different, and often embraces fundamentally different thought-forms, from the students' mother-tongue (17).

Related to the language-problem are questions of the role of geometry as a source of mathematical insight; does a shift from geometrical descripiton to symbolic presentation hinder the formation of concepts and thought-processes, and if so, in what ways? Another starting-point for enquiry is the development of probabilistic thinking. Finally, there is a need for far more research to find valid and reliable tests covering the whole range of agreed goals of the mathematics curriculum. The acid test of the importance of a declared goal is the account taken by society of the evidence of its achievement; but at the present time we lack the instruments to provide this evidence in respect of all but the most basic cognitive goals. This research is therefore an essential accompaniment of the programme put forward in this section.

NOTES AND REFERENCES

Notes

1. A useful, though somewhat out-of-date, reference source for Europe is Halls, W.D. and Humphreys, D., *European Curriculum Studies No. 1: Mathematics*, Council for Cultural Co-operation, Strasbourg, 1968.
2. The commission mentioned is "La Commission ministérielle sur l'Enseignement des Mathématiques" which was established in 1967 under the chairmanship of Professor Lichnerowicz. The quotation is from "Rapport préliminaire de la commission ministérielle", which is reprinted in *Bulletin de l'Association des Professeurs de Mathématiques de l'Enseignement Public*, 46th year, No. 258, 1967.
3. The document quoted is entitled: "Empfehlungen, Richtlinien und Rahmenpläne zur Modernisierung des Mathematikunterrichts an den allgemeinbildenden Schulen (Beschluss der Ständigen Konferenz der Kultusminister)". It is published e.g. in H. Schöne (ed), *Synopsis für moderne Schulmathematik*, Frankfurt (2nd ed.) 1974, p.172–182, and it has been reviewed in *Zentralblatt für Didaktik der Mathematik*, Vol. 2, No. 2 (1970), p. 42–43.
4. For particulars of the International Baccalaureate, which is accepted as a qualification for entrance to universities in many countries, reference should be made to the International Baccalaureate Office, Palais Wilson, 1211 Geneva 14, Switzerland. A brief account of the mathematical aspects of the curriculum and examination is given in an article by Morgan, J.B., in *The Mathematical Gazette*, Vol. 57 (1973), p. 183–186.
5. The problems of the mathematically gifted students are treated in a large number of publications. Some recent examples in which further references can be found are the following: Stanley, J.C., Keating, D.P. and Fox, L.M. (eds), *Mathematical Talent-Discovery, Description and Development*, John Hopkins UP, 1974; Krutetski, V.A., *The Psychology of Mathematical Abilities in Schoolchildren* (Trans. J. Teller), University of Chicago Press, 1976 (describes experiments and some educational practices in the U.S.S.R.); Erdös, P.. Child Prodigies, in *The Teaching of Algebra at Pre-College Level*, CEMREL, St. Louis, U.S.A., 1975 (describes case studies of gifted young Hungarian mathematicians).
6. Extensive information (with a very complete list of references) about mathematical contests is given in H. Freudenthal (ed), ICMI Report on Mathematical Contests in Secondary Education, *Educational Studies in Mathematics*, Vol. 2, No. 1, July 1969, p. 80–114. The problems given at the International Olympiads are now published by *Educational Studies in Mathematics* (see e.g. Vol. 6, No. 4, March 1976).

7. A few examples of students' journals are: *The Mathematics Student*, National Council of Teachers of Mathematics, U.S.A.; *Mathematical Spectrum*, Oxford University Press, U.K.; *Delta*, Popularny Miesiecznik Matematyczno-Fizyny, Warszawa, Palac Kultury, Poland; *Középiskolai Matematikai Lapok*, Hungary; *Trigon*, Mathematical Association of South Africa; *Mathmedia*, Institute of Mathematics, Academia Sinica, Nankang, Taipei, Taiwan; *Parabola*, University of New South Wales, Sydney, Australia.
8. The theme "abstraction" dealt with in subsection 3.4.1 is discussed in the North American context in the NACOME report, p. 14–22 (see the general references below). An interesting view ("that mathematics education would gain rather than suffer if it were agreed to drop the two words 'concrete' and 'abstract' from our vocabulary") is expressed by A.G. Howson (*The Mathematical Gazette*, 1974, p. 246–250) in a brief article "Giving meaning to mathematics", in which the author agrees with René Thom that "the real problem which confronts mathematics teaching . . . (is that) of the development of meaning" (Thom in "Modern mathematics: does it exist?", *Proceedings of the Second International Congress on Mathematical Education*, Cambridge U.P., 1973, p. 194–209). The question of giving meaning to mathematics was centrally placed at the Unesco Seminar on "Goals and Means regarding Applied Mathematics in School Teaching" (held in Lyon, 1974).
9. The Bavarian *Kollegstufe* is mentioned here only by way of an example. A description of the various aspects of this special programme (grades 11–13) is available in Kratz, *Der Mathematikunterricht in Kollegstufe*, Bayrischer Schulbuch-Verlag, 1972, of which a review is given in *Zentralblatt für Didaktik der Mathematik*, Vol. 6, No. 4 (1974), p. 165–167.
10. A good source of information about the IREMs is the *Bulletin inter IREM* (details are given in the references to chapter VII). The efforts of co-ordinating mathematics teaching and teaching of other subjects are mentioned in number 5 of the *Bulletin*.
11. Information about this project is given in Winter, K.L., The Continuing Maths Project, *Education in Science*, September 1974.
12. The philosophy of this project is dealt with in Ormell, C.P., Mathematics – applicable versus pure-and-applied, *International Journal of Mathematical Education in Science and Technology*, Vol. 3, No. 2, p. 125–131 (1972). Information about content and activities are provided in the "Students' Discussion Unit", see *Mathematics Changes Gear*, Schools Council, Heinemann Educational, 1975.
13. Reference may be made to the published proceedings of two recent conferences organized by the International Statistical Institute: Rade, L. (ed), *Statistics at the School Level*, Almqvist & Wiksell International, Stockholm, 1975; and Breny, H. (ed), *Statistics in Schools*, International Statistical Institute, Voorburg, Netherlands.
14. Four volumes have been published: Mosteller, F. et al. (ed), *Statistics by example*, Addison-Wesley Publishing Company, Menlo Park, U.S.A., 1973.
15. An interesting form of examination incorporating sealed hints developed in the Applicable Mathematics course (see section 4.1) of the Sixth Form Mathematics Curriculum Project is described in Chapter VIII, section 6.
16. These theories are outlined in an address to the 78th ASEE Annual Meeting and Conference, published in Gagné, R.M., Instruction based on research in learning, *Engineering Education*, Vol. 61 (1971), p. 519–523.
17. Many difficulties in the teaching and learning of mathematics are of linguistic type or origin. Such difficulties were studied at a Symposium in Nairobi, 1974, sponsored by Unesco, ICMI and CEDO. The origin of the Symposium was work in study groups during the Second International Congress on Mathematical Education. The report of the symposium (*Interactions between Linguistics and Mathematical Education*, Unesco, 1975) deals mainly with mathematics in general education. However, linguistic-related difficulties at later levels are closely linked to teaching and learning at the earlier stages.

General references

NACOME report, National Advisory Committee on Mathematical Education (NACOME), *Overview and Analysis of School Mathematics Grades K-12*, Conference Board of the Mathematical Sciences, Washington, 1975.

Schools Council 18+ Research Programme, *Report of the Mathematics Syllabus Steering Group*, Schools Council, 1977 (address given in references to chapter VII).

Unesco, *Goals and Means regarding Applied Mathematics in School Teaching*, Report of a Seminar in Lyon, 1974 Unesco, 1977.

PANEL

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Chapter IV

Mathematics education at university level

J. H. van Lint

INTRODUCTION

This chapter deals primarily with the education of mathematicians, although some trends related to disciplines which need mathematics as a tool are discussed in section 4. There is nothing about incidental mathematics courses which students in many different areas may take as options.

It is clear that suggestions which are useful in the context of one country are not necessarily useful everywhere. Nevertheless we hope that readers of this chapter will find something which they recognize as useful to them. If the information they subsequently require is not in the chapter (lack of space has limited me in mentioning sources) then they should write to me and I shall help if I can.

One extremely useful report should be mentioned right away, *Mathematics in India, Meeting the Challenge* (1973). A report concerning Europe, with nearly the same title as this chapter, appeared a few years ago (Fiala, 1970).

Some comments on the scope are necessary. In November 1975, an outline of this chapter was sent to 200 universities all over the world. Many questions were included with a request for answers and general information. The author is extremely indebted to those colleagues who took the trouble to send him answers, in fact often lengthy reports of a quality exceeding that of the present survey. Many of the suggestions were most helpful. This chapter was intended to be a report on international trends and I trust that many correspondents will indeed recognize some of their contributions and answers to the questionnaire in the following sections.

The final version of the chapter was written after the Third ICME in Karlsruhe and it incorporates a number of suggestions and corrections which were made by participants at the meeting and by an international panel of experts on the subject of this chapter.

I am most thankful for the international assistance which I received, and which helped me in taking personal responsibility for the contents of this survey. For the sake of information of the readers I mention the countries which contributed: Canada, United States, Costa Rica, Venezuela, Brazil, Argentina, Ireland, United Kingdom, Norway, Sweden, Denmark, Netherlands, Federal Republic of Germany, France, Switzerland, Italy, Czechoslovakia, Bulgaria, Libyan

Arab Jamahiriya, Egypt, Uganda, South Africa, Lebanon, Kuwait, Qatar, Bangladesh, India, Japan, Singapore, Australia, New Zealand. Besides these countries I mention Ghana, Israel and the U.S.S.R. because of contributions made at the Karlsruhe Congress.

Following this introduction, there are four sections in this chapter: 1. Curriculum trends (contents and objectives); 2. Structure of the programme (educational methods); 3. Mathematics as a minor subject; and 4. The role of university mathematics teachers. Within each section, we shall first discuss trends in objectives if they exist and then turn to trends concerning the actual achievement of these objectives. On that background, we discuss some of the corresponding problems and possible solutions for these.

1 CURRICULUM TRENDS

The education of mathematicians at bachelor's, master's and Ph.D. level has in many countries and during a long period of time essentially been based on the assumption that they would become research mathematicians, future university staff, or something comparable. Even the mathematicians who became teachers at secondary level or found employment in industry were often trained as research mathematicians. However, this has not been the case in all countries and now the pattern is changing everywhere, i.e. the objectives of mathematical education at universities are much more diverse than before. As an example we mention that the education at bachelor's level in many colleges in the United States has noticeably had as its objective the production of students who would be accepted by good graduate schools, while now a trend in the other direction – offering a balanced education leading to a good terminal degree – is starting. There is also increasing realization of the value of a bachelor's degree in mathematics as preparation for graduate work in fields other than mathematics. For data concerning trends related to undergraduate mathematics courses in the United States in the past five years, we refer to Botts and Fey (1976).

In a number of countries there now exist short terminal programmes; for example, 2-year courses for programmers and statistical analysts. A programme related to business and economics is described by MacDonald (1976). The development of more balanced programmes mentioned above, as opposed to the idea of specialization, is a trend which is being forced by the employment situation.

Another trend affecting the curriculum is caused by philosophical changes. In some countries, interpretations of the idea of "equality" of people have led to an egalitarian attitude which has a negative influence on the level of the curriculum.

A problem which has clearly not been solved, judging from the many situations that were described, is the idea of splitting mathematical education into several stages such that each of these is not only the qualifying part for the next stage, but also a satisfactory education to terminate with and then find corresponding suitable employment. In such a system, the first level would cover basic mathematics and specialization could occur at subsequent levels. There is some experience with this idea in Bulgaria (Iliev, 1976).

1.1 Early years – e.g. leading to a bachelor's degree

We consider below problems relating to the education of students entering the university, after some kind of secondary education, with the intent of majoring in mathematics.

1.1.1 *More abstract courses*

About 20 years ago, Bourbakism started to have its effect on the first years of mathematical education. Traditional calculus and geometry courses were replaced by more abstract courses. As examples we mention introduction of general topology and metric spaces at a very early stage of the curriculum. Even abstract integration and Lebesgue integration in the freshman year has occurred. The most significant change was the disappearance of geometry which became completely algebraic (linear algebra, etc.). Probably this process started in France where the introduction of abstraction in the freshman year was easier because of generally higher level of the students entering university. For a long time one could say that increasing abstraction was a trend almost everywhere and simultaneously an ever increasing stress on "rigour". From the answers to our questionnaire, it seems that there are only a few countries where the process has not started yet (we know of Japan and the U.S.S.R.). Many correspondents report that abstraction is still increasing. However, from the majority of the replies the impression emerges that the most general recent trend is a slowing down of the process and in many countries a definite reversing of the tendency towards rigour and abstraction. Complaints about lack of motivation for the mathematics which is taught and not enough intuition being used in the courses are general.

It is of course hard to measure the effect which this abstraction has had. However, there is a feeling that it has led to a compression of material, making it harder for students to absorb the course content and to relate it to concrete problems, to examples in physics, etc. It is significant that most universities in the United Kingdom and the United States feel that we have overreached ourselves. The effect has been that some students have learned to play the game, with little idea why, and that the weaker students were completely lost.

This is an area where international discussion is desirable. The countries where certain mistakes have not been made yet would certainly profit! Clearly the language of metric spaces and the methods of linear algebra have been an improvement and they will remain, along with many notational improvements. It would be wise to find a balance between those new methods which have generally been considered succesful and the topics in concrete analysis, geometry, physical applications, etc. which disappeared but should not have done.

1.1.2 *New subjects*

Besides the change in style treated in the previous section, another significant trend has been the introduction of new subjects in the curriculum for the early years. In the first place, we mention statistics which has made its entry practically everywhere. In some countries, the change has been easier because probability theory was introduced in the secondary school programme (although there is some doubt whether the pupils really learn much about probability theory). A recommended programme can be found in *A Compendium of CUPM Recommendations*, Vol. II (1973). In some places, topics from operations research have also been introduced.

Next on the list are subjects from computer science (or informatics as it is now called). Both these and probability theory mentioned above are usually compulsory subjects. As far as computing is concerned, the minimum is usually a course on programming. Other topics are algorithms, languages and elementary numerical analysis. We refer to *A Compendium of CUPM Recommendations*, Vol. II, p. 571–627, where the main theme is the introduction of computing aspects into many different topics of the curriculum.

A more recent trend is the introduction of a selection of topics from discrete mathematics in the early years (e.g. the United States, eastern Europe, Netherlands, Denmark). There are many reasons for this. Clearly there are strong connections with applications, statistics and computing.

Another important point is the possibility of offering a challenge in a programme that might otherwise bore or scare off many students. Most important of all is probably the possibility of rediscovery (by the students) of results, which has become rather unlikely in the more abstract parts of the curriculum.

Of course, the coming of new subjects has implied the disappearance of others. One of these is geometry. From many responses it has become clear that practically everywhere the amount of physics which is compulsory for students of mathematics has decreased very much, sometimes to nothing! In the long run this may prove to be a most unfavourable development. In some universities, extended courses in the new topics treated in this section are options. They seem to be replacing courses in hard analysis and other difficult subjects, thus creating the danger of unbalanced education and too early specialization.

As developments in the near future we expect, an increase in the number of computing-oriented calculus courses (cf. *CUPM Recommendations*, Vol. II), more emphasis on the algorithmic approach to mathematics (cf. Chapter XIII), and a growing influence of hand-held calculators on the curriculum, probably also in high schools (National Science Foundation Report, 1976). When considering such trends and maybe even more such innovations, it is essential that we keep in mind that basic analysis, linear algebra and elementary probability theory remain the most important subjects to teach and are usually more than enough for most students to digest.

1.1.3 *Level of students at entrance*

In this section we examine trends related to the level of students entering the university. Mathematical education at primary and secondary level has been changing much more in recent years than at the university and the effects of this change are being felt. Some sources observe an increase in the ability of young students to handle abstraction (whatever that may be worth; cf. Chapter III, section 3.4). In some countries, the introduction of probability theory in secondary schools has had favourable effects. Certainly the early use of linear algebra is considered advantageous, but the toll that had to be paid was too heavy. From all corners of the world one hears that because of the replacement of Euclidean geometry average students now have practically no geometric knowledge, intuition or insight, and as a result one has to devote more time to trying to help students to develop this insight.

There were some correspondents who observed that in their country the goal of secondary education was not to prepare for tertiary education but to prepare for life as a citizen. Often this is a reaction against past university dominance of final year secondary school curricula. In countries where this is true, the preparation for tertiary education apparently becomes a preliminary task of the universities!

It was the opinion of some of the participants at the Third ICME that even if present day secondary education is a less suitable preparation for a university education in mathematics, it has become a better education for the majority of the pupils (who do not go on to study mathematics).

Practically universal is the complaint about a most significant trend: the strong decrease in manipulative skills of students entering university. The same is true for other fundamental skills such as reading and writing. (Older people claim that this complaint has always been made!)

Not always, but certainly quite often, the "new math" is described as a complete failure, partly because too much change took place too quickly after decades of immobility, partly because the teachers were not prepared for the change and there was not enough suitable material available (see Chapter VIII, section 2.1). Data to support opinions are known (e.g. a project by

Dr. J. Hunter of the University of Glasgow to measure the state of preparedness of students at entry to the university). However, in the United States new math has been around for about twenty years. The recent sudden drop in preparation of first year undergraduates cannot be due to new math alone. Probably the causes are of a social nature. It is very important to find out what they are.

Besides the change of mathematical material in secondary schools, many new subjects have been introduced, resulting in a decreasing number of classroom hours for mathematics (in several countries). In addition to a decrease in manipulative skills it seems that students are also less able to handle problems including several steps. In many universities, it has become necessary to use remedial teaching for the freshmen. In others, which have not reacted yet, it is feared that soon the standards will have to be lowered. It is not unreasonable to assume that one of the most important factors influencing the level of the freshmen is the tremendous increase in the number of students entering university in recent years. First of all, this means that the social and cultural background of the present day student differs from what it was.

This implies an increased educational task for the universities. Furthermore, many students are far less motivated than they used to be and many are absolutely unable to complete the traditional university education. Consequently, considerably more staff energy goes into guidance and counselling than into the teaching itself. It is probably advisable to have an examination at the end of the freshman year and to structure the curriculum in such a way that the first year gives a good indication of the probability of completing a mathematics education in a reasonable time. At least this is better than having an entrance examination for possibly insufficiently prepared students. It is clear that if one follows this suggestion, a high dropout rate after one year should be considered acceptable.

Besides the remarks made above, it is a personal impression of the author that every year students are less able to express themselves reasonably in their own language (or any other), which clearly has a disastrous effect on the ability to study mathematics.

One can hardly expect an easy solution for the problems of this section. The topic "new math" alone is enough for lengthy arguments. However, some of the facts mentioned above, observed by many authorities, make it clear that in the future there should be much more contact between secondary and tertiary education concerning programmes and methods. We refer to a recent report by the German Mathematical Society concerning insufficient preparation of undergraduates in mathematics (*Deutsche Mathematiker-Vereinigung*, 1976). The fact that the expression "remedial teaching" was used by a number of correspondents shows that there is not enough agreement on the division of tasks between secondary and tertiary education (see also Chapter III, section 3.6).

At a meeting on the teaching of mathematics at university level, held in Strasbourg in 1969, one of the conclusions was that pre-university education for those who wish to take up university studies in mathematics should include some linear algebra, elementary functions and some calculus, the terminology and notation of set theory, the notion of a group and group isomorphism through examples and some combinatorial calculation (Fiala, 1970). I believe that it should be made clear that this list should also include geometric insight (and it should be specifically stated what is actually desired in this respect), manipulative skills and maybe some elementary probability theory. Even without those, it is doubtful whether the recommendations of the Strasbourg meeting can be achieved in present day secondary education.

Concerning the level of students entering the university mathematics departments in developing countries, there is a trend that should be mentioned. In many of these countries, the importance of mathematics is not realized sufficiently by the authorities and consequently the career possibilities for mathematicians are not as appealing as those in many other fields (mathematicians receive low salaries). The level of the freshmen reflects this undesirable situation; changing it is strongly advised!

1.1.4 *The non-mathematical content of the programme*

It is much more difficult to identify general trends in the non-mathematical content of the undergraduate programme. There have always been tremendous differences from country to country. Language requirements were absolutely necessary (as a preparation for future study) in many countries and this remains true, whereas in others the necessary knowledge of languages is obtained through secondary education.

Also, the objectives of undergraduate education still vary from a preparation for further education in mathematics to a broad general education with an accent on mathematics. In general, the English-speaking countries still have the broadest programme, whereas many European countries have curricula including only mathematics, physics and mechanics.

Nevertheless, there is a general trend towards a decrease in the number of compulsory physics courses and also in the number of physics courses which are chosen when they are optional.

In some universities, the increase in the number of students has resulted in parallel courses, e.g. a course on physics especially for mathematicians (instead of both mathematicians and physicists). Strangely enough, there are many many complaints that this has lowered the level of teaching of physics for mathematicians. Such facts, and the presence of many new areas where mathematics is applied at a reasonable level, e.g. economics, have led to a decreasing appreciation, by both students and faculty, for physics as part of a mathematics education. Many universities report that physics has completely disappeared from their mathematics programme. A few mention that as a reaction to the over-abstraction of a few years ago mechanics and physics have been reintroduced as compulsory subjects.

A second general trend is that the number of non-mathematical options has increased. Again this varies quite a lot. Sometimes it means courses in economics, computing, general science, etc. In other cases it is broader, including subjects like geography or geology.

But often, e.g. United States, United Kingdom, practically anything seems to be a possible subject. In these countries, together with a larger variety of options, there is a decrease in the number of compulsory courses.

If we look at the countries where a bachelor's degree is only a preparation for graduate study and not a degree with which one can terminate the study, there is a definite trend to have less time available for non-mathematical parts of the curriculum. Partly, these have had to make way for statistics and computing, just as some parts of mathematics itself have had to be dropped.

1.1.5 *Mathematical models*

Our next subject is connected to the previous one. It is not a widespread trend yet, but starting slowly in a number of countries (e.g. United States, United Kingdom, France, Netherlands, the Scandinavian countries), courses in the construction of mathematical models are being introduced. (We refer to Chapter XII, and to Maki and Thompson, 1973.) Usually there are several motivations for this such as the decrease in physics courses, the increasing number of disciplines in which interesting mathematical problems arise, the fact that only a few of the students will become academic mathematicians, etc. Generally the courses emphasize the principles behind the construction of mathematical models, and how a model should contribute to the understanding of the real situation it is intended to describe. It is recommended that the problems should be chosen from a wide range of different subjects. There is not much experience with these courses yet, and on the whole this is a difficult theme to teach. A lot depends on non-mathematical questions, such as good judgement and understanding of the practical situation. Furthermore, if one wishes to teach such courses early then one is often faced with the problem of not having enough mathematics available to actually solve the mathematical problem. We expect that the number of courses of this type will increase in the future. Designing them is a lot of work. To avoid unnecessary duplication of this work, it is desirable that course contents, results, classroom experience, etc. are published (e.g. in *Educational Studies in Mathematics* or the *International Journal of Mathematical Education in Science and Technology*). For a report on this subject, we refer the reader to § 3.3.2 of *Adapting university mathematics to current and future educational needs* (London Math. Soc., 1975). Also see the *CUPM Recommendations* (1973), Noble (1967) and Gross (1972).

1.1.6 *Mathematical engineering*

For more than ten years, there have been institutions in some countries offering a programme of mathematical engineering, e.g. the technological universities in the Netherlands, in Darmstadt, Vienna, Graz, some of the *Ecoles d'Ingénieurs* in France, some institutions in the United Kingdom and Sweden. Since the phrase "mathematical engineering" is probably unfamiliar, we shall define it in this section. First of all, we do not identify it with applied mathematics (Seidel, 1973, and *SIAM Review*, 1975). Neither are we thinking of short programmes in industrial mathematics with internships in industry (sandwich courses).

The original questionnaire which was sent out to prepare this chapter contained a question asking whether the curricula at universities and technological institutes were growing further apart (in the Netherlands, Denmark and Argentina the answer was yes; in France the opposite is the case). The question caused a lot of confusion since there is a tremendous diversity in technical institutes, technological institutes, etc.

In this chapter we mean by *institutes of mathematical engineering*, institutes which offer an education in technology at the university level. Many countries have technical institutes of a considerably lower level than universities and the point of the question was lost. In some of these countries, the technical institutes were imitating the university curricula in an attempt to improve their image. Still others did have technological universities but their mathematics departments offered the same education as universities.

Somewhat closer to what is meant by mathematical engineering is the possibility of specializing in mathematics offered by some engineering and physics departments (e.g. Sweden). I am convinced that for the future it is most important that in every country there should be a possibility to obtain an education in mathematical engineering. Certainly for the developing countries this is essential, but it is equally important in the countries which have been producing too many potential academic mathematicians. Finally, future teachers of mathematics will probably do a better job if they have been educated with a good background in applications.

The main principle behind the term *mathematical engineer* is that one must learn to understand the problems of non-professional mathematicians, find solutions and then explain them to the customers (McLone, 1973). A long and extremely useful discussion of what one wants can be found in "Education in Applied Mathematics" (*SIAM Review*, 1967); here, however, the theme was still applied mathematics. A more accurate description of what is meant, including a possible programme, can be found in "Mathematical engineering, a five-year program", p. 649–683 in the *CUPM Recommendations*.

An important problem to be discussed in connection with this different curriculum is how to inform both those teaching mathematics and those giving advice on careers what the mathematical engineering programmes are. Even today it is not uncommon for a secondary school teacher in the Netherlands to advise a pupil interested in mathematics not to go to a technological university because no "real mathematics" is taught there; this seems to happen in other countries too.

1.1.7 *Students' opportunities for choice*

What are the trends regarding the amount of choice which students have when making up a programme? In preparing for this survey, it was my opinion that this has been increasing. The basis of this opinion was knowledge of programmes of some years ago involving practically no choice whatsoever, and information on recent local trends.

From most of the correspondents I learned that in the early years there generally still is very little choice. The institutes where the early years are preparatory usually offer the possibility of choosing from a few, otherwise practically fixed, programmes. In many universities in the United Kingdom and the United States (more generally in those countries where one can terminate with a bachelor's degree) there is indeed more choice than ever. From some of the correspondents I received the impression that this had gone too far, leading to extremely unbalanced programmes and superficial development. There is very little point in studying to become a dilettante in three different subjects simultaneously. Generally, the system of degrees based on a number of credit points only did not meet with much approval.

A problem we face in the coming years is increasing pressure from students for more choice and options, supported by members of staff who wish to teach their own specialism. Concerning this problem, there are two recent trends. First of all it is clear that in the past years there has been an enormous amount of thinking and talking about curricula meeting the needs of society and of the students. At the same time we observe the trend that more and more students (even among those who do not know any mathematics yet) seem to believe that they know better what they should learn than do their educators. In my opinion it would be irresponsible to give in, even though it would certainly be easier and save us a lot of time!

1.2 Later years – leading to something like a master's degree

In this chapter, the treatment of the education of mathematicians after the bachelor's degree has been split into a section on the master's degree and one on the doctorate. In some cases, this does not fit the actual situation, e.g. sometimes the Ph.D. follows after the bachelor's degree. That this is becoming less common in the United States can be seen from a report in *Notices of the American Mathematical Society* (1976). If the Ph.D. follows after the bachelor's degree, I would consider the course work required for the Ph.D. as comparable with the master's education in those countries where this is the common terminating degree. In section 1.3.1 below on Ph.D. programmes we shall be interested in the thesis.

In the master's phase we consider three qestions: first, the degree of specialization and corresponding trends; next, the interaction of mathematics with other fields, – at this stage it is no longer a question of motivation for doing mathematics but a question of true applications of advanced mathematical techniques to other subjects; and finally, we raise the question whether at this stage of education the student is actually preparing for some specific function as a mathematician in society.

1.2.1 *Specialization*

The question of specialization in the circulated questionnaire has not produced replies of very much interest. Many universities answered that indeed specialization at this stage was common, notably so for the typically vocational directions such as statistics and computer science. It was not unusual that the degree of specialization depended on whether a Ph.D. programme existed or not. In the former case, one tried to postpone specialization; in the latter, specialization at master's level was unavoidable. No trend can be discerned. It is true that in countries where the employment situation gives cause for concern the question has been asked whether a more general education would increase the possibility of finding employment after graduation. However, this has not led to any action yet.

1.2.2 *Interaction between mathematics and other sciences*

Traditionally physics has been the subject where students of mathematics could study the interaction of mathematics with other sciences. Here, high level mathematics is used and quite often the mathematics was even developed to handle the physical situation. Departments with a strong applied mathematics group still stress such applications. However, the decreasing knowledge of physics which the students have is making this more difficult. Many departments mention both biology and economics as subjects which are playing an increasing role in the curriculum at this level, and some already have research groups in one of these areas. Quite often the final year of study is spent on a joint project with a biology or economics department. Future developments in this area look promising. Whether catastrophe theory falls under this heading is not clear, but it was mentioned a few times.

There are departments of mathematics which have established contact with industry, seeking to identify industrial problems where mathematicians can help or where interesting mathematical problems arise. This is a trend to be encouraged but it will take some time to develop. It certainly would be valuable for students to participate in these activities.

With respect to this question, control theory was mentioned a few times. Probably this is considered too mathematical to fall under the present heading. The social sciences other than economics were not mentioned at all. Clearly, the mathematics used in this area is not of a sufficiently high level to be able to contribute to the mathematics curriculum at graduate level.

If we consider this part of the curriculum of importance (and the author certainly does), then there is a big problem to be solved. Smaller universities and also the universities in the developing countries often do not have staff in the mathematics department with any knowledge of subjects such as biology or economics. (In larger universities, joint appointments have proved to be quite succesful.) There is very little literature which would make it possible to organize seminars on these topics. How can one, in such circumstances, introduce such topics? It may interest the reader that some of the large departments of mathematics admitted that very few of their staff knew even any physics!

1.2.3 *Preparing for specific functions in society*

Is the curriculum designed in such a way that in the final part of the course the student is actually preparing for some specific function as a mathematician in society? Let me start with the developing countries where this is a very important question and where the danger exists that universities stress some subjects unwisely. Sending the students abroad for the final phase of education may be questionable, since it is essential to give them motivation and opportunity to serve in their own country after graduating and this indicates that they should remain there. The project known as *Centre Pédagogique Supérieur de Bamako* in Mali is apparently succeeding in this respect. We refer the interested reader to a report on this project (D'Ambrosio, 1975). We also remark that it is not very stimulating for mathematicians in developing countries if their countrymen who finally return home after a prolonged stay abroad get appointed to higher positions than those who remained in the country to do a lot of very necessary work!

With the exception of eastern Europe, there seems to be very little attention paid to the future functions of the students in the design of the programme. An exception is of course made for the vocational directions such as statistics and computer science. Clearly the mathematical engineering programme mentioned earlier is specifically designed to create mathematicians who will work in industry. On the other hand, some countries have practically no preparation for mathematicians in industry. A common answer to my question was that there was such a diversity of careers that the only sensible thing to do was to teach mathematics in its own right. A good mathematician can learn the rest in a preliminary period of training in his chosen career. One trend mentioned was stressing the necessity of learning how to write about project work. This is useful in many future careers.

Although the education of future mathematics teachers was originally excluded from this chapter we cannot ignore it completely. It is my strong opinon that the preparation of teachers for the highest secondary level definitely belongs to this chapter, because I feel that they should receive a rather large part of their education together with future research mathematicians and industrial mathematicians. However, a student intending to become a teacher should also prepare for *this* function while he is at the university. Therefore the problem is related to the present section and a question concerning the education of future teachers was included in the questionnaire. In some countries, secondary school mathematics is not taught by university graduates (a deplorable situation) and hence the question was left unanswered. General agreement with my point of view (reflecting the situation in the Netherlands) came from the United Kingdom. Here the attitude was also that intending teachers should study mathematics deeply and widely and should be familiar with its place in society as a whole. In some British universities special programmes exist (Howson, 1975).

It is the opinion of the author that future teachers should have a course on the language and the structure of mathematics which should be taught by a first rate mathematician. Essential questions such as "What is a variable?" and "What is an axiom?" should be treated in the course. Also, the recent history of mathematics should be treated in a special course. In any case, some experience with research (at any level) is necessary.

A notable programme preparing students for a specific function is the Doctor of Arts programme (cf. section 1.3.1 below). This includes topics in the teaching of undergraduate mathematics.

1.3 (Post-) graduate – doctorates and later

1.3.1 *The Ph.D. programmes*

Regarding trends in Ph.D. programmes, there is much to report, including some quite interesting developments. Before considering these, we discuss the situation where no Ph.D. programme exists. There are several reasons for this. In Norway and Denmark, there is a doctorate of higher level than we mean by the Ph.D., and the work is done without supervision. In Sweden and Denmark, there exists a new doctorate at the Ph.D. level. The purpose is to facilitate the entry of graduates into the job market. In Italy, there is no Ph.D. programme. Many smaller universities and at present many universities in developing countries do not offer such a programme, and as a consequence, students often go abroad to obtain a Ph.D. For other reasons it also happens (e.g. in Europe) that students acquire their degree in a foreign country. Besides the obvious language problems, different entrance standards, prerequisites, etc. make this switching from one university to another much harder than it should be in this age. A suggestion worth looking into is the following. It should be possible for a university in a developing country to establish a form of co-operation with a university in a foreign country (for example in Europe). This co-operation could be structured in such a way that it becomes easier for the host university to accept students from abroad, and to arrange for the home institution to award the degree when the student has reached a level which is locally acceptable. Certainly there is no reason (at present) to insist that the Ph.D. degree should have the same level everywhere.

The situation concerning the Ph.D. production in the United States and Europe has been quite different, but in both cases it has led to new ideas about the programme. The United States has had an overproduction of Ph.D.s. This overproduction was caused by the fact that during a long period the number of students increased rapidly and there were not enough Ph.D.s to fill the teaching positions. This has now reversed. A resulting trend is a tendency to make the programme broader to try to produce employable mathematicians. Certainly the percentage of Ph.D.s. in applied mathematics is increasing.

In many European countries it has always been difficult to get a Ph.D. in mathematics, certainly if one compares with other disciplines (e.g. in the Federal Republic of Germany less than 3 per cent of the master's degrees go on to the Ph.D.). In France the level is even going up. In a number of countries the level is being reconsidered and requirements are being slightly lowered.

A considerable piece of original research remains the main requirement, but it has become possible (e.g. France, Netherlands) to get the degree on the basis of a collection of papers written in a period of less than five years to which an introduction has to be added. In other countries, similar systems are sometimes used (even without the time limit), in some cases including compulsory course work.

Another trend seems to be that the value attached by society to a Ph.D. degree is decreasing, with the exception of academic circles. It is claimed by some people that it is harder to find certain kinds of employment (such as a teaching position in a 2-year college in the United States and certain positions in industry), if one has a Ph.D. degree than without it. It would be most undesirable if it became a general attitude that a Ph.D. had no other purpose than to improve one's prospects in teaching. An important development about which much cannot be said in the available space is the American Doctor of Arts degree. The degree was proposed in the early 60s (cf. E. Moise, 1961) in the period when the Ph.D. production was too low and the creative requirements were too high for many students. It was intended as a difficult course of study for which success would be reasonably predictable for suitably qualified people; (this is not the case for the Ph.D.).

The essential idea was for the new degree to represent scholarly achievement, but not necessarily creative talent. The programme was designed to provide exceptionally strong preparation for teaching mathematics to undergraduates. This programme merits a serious study especially by those countries which have a demand for teachers of mathematics (Bushaw, 1973). The idea of a thesis which has little original work but contains a good survey, research in recent history of mathematics, some didactical problem, etc., which is included in the above programme, does not seem to have been followed anywhere else yet. Clearly serious consideration is necessary before one starts such a programme. There is a rather large probability that the D.A. becomes a degree for those who tried for the Ph.D. and failed, and this should be avoided. A D.A. programme can succeed if the Ph.D. programme is taken very seriously, if there are enough people of real mathematical talent to fill the available positions, and finally, if there are people who can take on the difficult task of supervising D.A. theses.

1.3.2 *Continuing education*

Is there a visible increasing task for university departments of mathematics related to continuing education? (See Chapter V.) The need is felt and has been officially acknowledged by a number of governments (which generally do not subsequently provide the necessary funds). In Czechoslovakia, legal measures in this direction are expected. In Sweden, more continuing education at the universities is expected. There are several countries in which universities have offered refresher courses for teachers and often there are institutes (like IREM in France, IOWO in the Netherlands, the Open University in the United Kingdom) which organize this on a national scale. It is said that the effect has often been disastrous because the university lecturers had no conception of how much the teachers had forgotten, how quickly they could assimilate new work, what they really needed, and how it should be taught. In France this is no longer true, showing that it is possible to learn how to do the job in the right way.

Clearly this is an area where a lot of thinking still has to be done. The introduction of computing aspects in secondary schools is a separate problem (IFIP Report, 1972). Refresher courses for scientists and engineers are being discussed in many places, but university experience is limited. Some industries (e.g. Bell Laboratories) offer a continuing education programme for their own employees, often including courses in mathematics at all university levels. Several universities in the United States (e.g. University of California in Los Angeles – UCLA) have offered such courses for a number of years (see Chapter V).

2 STRUCTURE OF THE PROGRAMME (EDUCATIONAL METHODS)

When considering objectives related to structure, the question is not what mathematics one is teaching or why, but what one is trying to achieve through the special form of the programme. For instance, a series of lectures can have as objective the complete teaching of a subject or only the preparation of the students for reading the literature (thus learning the subject on their own), or anything in between. Objectives related to the educational methods of the programme are motivation, enthusiasm, developing technique versus understanding, coping with different intake levels, etc. We have tried to discover the trends in this area.

2.1 The courses – form and style

One cannot say that there are many changes in courses. Complete explanation is more common than global treatment and usually problem sessions are connected to the courses. Books are used as often as not. There is a trend to use duplicated lecture notes more than before, with the effect that students do even less reading around the course on their own than before. Some of the courses are including more material to motivate students than a few years ago. Representing opposing trends are the integrated courses given in some universities to show connections between various topics, and efforts by others to make it easier for students by breaking up the subjects into labelled bits. In my opinion, this last mentioned strategy has a bad effect on students who often have a tendency to make such breaking up on their own initiative and to regard the bits as independent and unrelated.

Some experimentation has been going on with "self-paced courses". One of the purposes is to cope with the problem of diversity of background of freshmen. The method is said to be suitable for courses with a high manipulative content. There were also some reports of successful use of the method to introduce topics such as elementary linear algebra, in this case alongside traditional lecturing methods. I have seen very little of the method and I must admit that I have strong doubts about its value. An explanation of the nature of these doubts and support for them can be found in Moise (1973). Some information on an experiment in Southampton can be found in §2.3 of the Proceedings of the Nottingham Conference (London Math. Soc., 1975).

An idea which has been used in a number of places, not always successfully, is the "problem clinic", manned by staff and senior students for several hours a day, where students can drop in with their problems.

Several audiovisual expedients are now being used. We mention videotape recordings of lectures which make it possible for students to see and hear certain difficult parts of a course for a second time and which also offer the possibility of having lectures, held by a visiting expert, repeated annually.

2.2 Apprentice-type education

A new trend, not widespread yet, is the method of apprentice-type education. There are several variants. One is to have the student work for a short period of time in one of the research groups of an institute and, depending on what is going on at that time, acquiring the background knowledge which the researchers already have, by experiencing the need for this knowledge and finding it in the literature. Active participation in the work of course also helps.

A second new idea is the project group in which the whole group starts more or less from scratch with the purpose of solving some open problem. The idea is that the members of the group will be motivated by the problem to acquire the necessary knowledge and skills, but despite the many slogans used by advocates of the method it seems to the author to be just another example of misplaced egalitarianism, and clearly to be an ineffective and extremely time consuming approach. Reports of failure of the method are hence not surprising to the author. A more structured attempt to use the idea of project work as part of the course is the educational experiment at Roskilde (Denmark). We refer the interested reader to Niss (1976).

Extremely useful, especially for the applied mathematicians, is an industrial internship as part of the programme (cf. *The Statistician*, 1976). For this, one needs good contacts with industrial laboratories. The method is being used successfully in a number of countries. (Regarding an industrial internship as part of the education of mathematics teachers, we refer to section 3.4 of Chapter XII.)

The idea of an internship in one of the research groups of an institute is not being used as much as it should. Many research projects which are funded by grants hire students as assistants, e.g. in vacations, but this is not what is meant. The internship should be a recognized part of the curriculum (as it is at my own university).

The idea of an external (teaching) internship is part of the D.A. programme at Washington State University where it is considered an essential element. Both a teaching and a service internship (in industry) are a useful aid in the matchmaking process between prospective employer and prospective employee.

Several American universities have an "Undergraduate Research Opportunity Program", where undergraduates may work alongside some faculty member or outside scientist in their research. (Massachusetts Institute of Technology – M.I.T. – reports that the mathematics department does not have much experience with the programme yet.)

2.3 The role of problem solving

Problem solving is one of the essential activities in mathematics. However, the period of abstraction seems to me to have de-emphasized this fundamental fact. In the questionnaire a question was included whether problem solving is ever a recognized part of the curriculum. Here I do not refer to the problem sessions corresponding to certain lectures but e.g. to special problem solving seminars (we have a Polya-Szegö seminar for students) or groups (including students) which regularly meet to solve all kinds of problems (e.g. from the problem sections of journals). To me there is little doubt that this could be an important contribution to the education of most students. Many of the replies received came down to "A very nice idea; too bad that we do not do it at our university". A positive response was received from Czechoslovakia. Problem solving seems to be in bad shape in many countries, and a trend which was pointed out is that students find problem solving too hard and prefer proving various trivial consequences of axioms and definitions. A number of American universities have a tradition of intensively coaching the best students for the Putnam competition, but this is an extra-curricular activity. From the correspondents I believe that a large number of universities feel that a revival of problem solving should be encouraged; a weak but promising trend.

2.4 The role of research

The next question we look into is whether some form of research plays a role in the educational programme up to master's degree (for the Ph.D. it obviously does). For the master's degree it is customary in some countries (examples are Australia, the Federal Republic of Germany, the United Kingdom, Netherlands, Czechoslovakia) to require a thesis or an essay or a report.

This thesis takes from one half to one year of research, often guided. New results are generally not demanded, but the work must be done by the student. Sometimes a survey of some area is also satisfactory. It is significant that in those universities where this system is used it is considered one of the most essential parts of the course. The idea of research projects at undergraduate level was more or less new to most correspondents. I have used it with very satisfactory results for discrete mathematics courses in Eindhoven. This is an area where individual enterprise is possible and indeed some of the students have produced results which were subsequently published in journals. Even if the results are not that good, it is a rewarding experience for the students after so many years of just learning known results. The same method has been used at Southampton (see section 3 in Howson, 1975, and Hirst and Biggs, 1969). Here too, the results were very good. It seems worthwhile to suggest that others experiment with these ideas (also see p. 95 and p. 117 of *The Statistician*, 1976, and Bajpai et al., 1976).

A difficulty with these projects (observed both in Southampton and by myself) is that many students tend to become too committed, to the detriment of other work. It is important that the supervisor carefully checks scope and demands of the project and that appropriate credit is given. Since some of the projects misfire, there should also be a possibility for the supervisor to terminate the project (again awarding appropriate credit for the work which was done).

Some universities have "elementary seminars" for undergraduates. This does not seem to be much of a success. The students preparing for their talk learn something and possibly the presentation itself is a useful experience but the effect of having to listen to these presentations is more or less disastrous. However, this could be a good idea for intending teachers. A much better idea came from Czechoslovakia. Here students' research groups present their results at annual meetings of the students' branch of the Socialist Youth Association. The fact that the best papers are awarded prizes stimulates the students.

2.5 Examination methods

We have tried to identify trends in examination methods (see also Chapter VIII). There seem to be few. One of them is the introduction of frequent smaller tests during courses, the results of which are used as part of the final assessment. More generally, one can say that the idea of continuous assessment is becoming more popular. In fact, despite an increase in the number of students, there is more assessment by observation than in the past. A second trend, probably also related to large numbers of students and pressure to study quickly, is the tendency of students to take examinations in subjects much too soon after completion of the course. In this way there is no time for a deepening of understanding nor for practice in manipulation.

There is also a greater variety in types of assessment (e.g. as indicated in section 2.4) but multiple choice tests have practically disappeared in many countries, and this is in my opinion a fortunate development. One effect of experiments with multiple choice tests and similar methods is a trend away from long answer tests to short answer tests, which share some of the advantages of multiple choice tests but which are not as difficult to design properly.

3 MATHEMATICS AS A MINOR SUBJECT

It is not possible to do justice to this important topic in a few pages. We limit ourselves to a few observations, pointing out recent trends in objectives and curriculum structure. If we first look at traditional service tasks such as teaching the necessary mathematics to physicists and engineers, then already there are marked changes. The problem of motivation is taking too large an amount of the available time. From an interesting paper by H. Halberstam (1972) I quote the following remark concerning the idea of just trusting the teachers: "... that would be reckoning without that grim, humourless rationalisation of the unwillingness to learn that is characteristic of so many students of our day. ... they all ask the same question, 'What's the use of mathematics to me?' ..."

This problem of motivation is becoming more difficult for a second reason. As we have already discussed above, more and more members of the faculty of mathematics departments know hardly any physics themselves. One fears that the argument runs: "I teach them mathematics; they'll find out for themselves when and where they need it". Since there is often very little time to teach the students the manipulative skills which they must have, such a remark would be easy to defend but as Halberstam pointed out it is no longer easy to put into practice. For an interesting survey of the many problems in this area and a number of suggestions for improvement we refer to Bajpai et al. (1975). It seems that many of the problems discussed in this section do not exist in the U.S.S.R.

A very clear trend is the fact that more and more disciplines need mathematics and as a consequence service courses in mathematics for the social sciences, economists, psychologists, and recently also for biologists are appearing all over the world. Here too, the motivational aspect is important but there is another problem. It is not uncommon for the departments in question to have large numbers of mediocre students. These students take these mathematics courses at an early stage of their studies and there it becomes one of the principal obstacles. Clearly such courses are not too popular! Looking at the contents of the service courses, we can say that the stress is still on manipulative skills and motivation. Many universities remarked that the mathematics courses for engineers and physicists have recently been getting more higher level mathematics put into them. Other important changes are: (a) an increase in the amount of linear algebra taught, particularly an increase in the amount of and emphasis on matrix theory; (b) a reduction in the amount of geometry taught; (c) an increase in the amount of probability and statistics taught. Concerning courses for the social sciences we refer to *CUPM Recommendations* (1973), and Selby (1973).

The courses for biologists seem to be rather successful and not unpopular. The important thing in such courses is clearly not to try to transform the students into active mathematicians but to show them which problems in biology can be attacked with mathematics and what kind of mathematics is used. It is sufficient if in their later careers they recognize such situations and then know who to consult with their problem. A typical course (Van der Blij, 1974) contains some elementary calculus and differential equations applied to growth processes, elementary probability theory, normal distribution, the concept of linearity (mappings, etc.), some linear programming, approximations to functions, diffusion; the examples (and pictures!) are chosen from biology. It would be a very good idea to collect the material from such courses (which have apparently been given at many universities in the last few years) in one place (where and how?) so that others can profit from the groundwork which has already been done.

We point out another trend. It hardly ever happens any more that topics suitable for physicists or engineers and also for mathematics students are taught to these groups united in one class. Thus the technological university in Trondheim reports that a topic such as functional analysis is typically offered in separate versions. This seems unfortunate to me. A dangerous trend and one which should be discussed seriously by the mathematical community is the increasing number of mathematics courses (for non-mathematicians) being offered by other departments. A good example is the striking number of poor statistics courses being taught by engineers, social scientists, etc. Most professional mathematicians would be a lot more hesitant about teaching such a special topic if they were not experts themselves! If the mathematicians believe that they should halt and reverse this trend (and I surely do), then they should first find out which mistakes have been made in the past. It is essential that the lecturer be sympathetic to the needs of the students and aware of their difficulties. A very good suggestion, put forward in connection with the questionnaire by E. M. Patterson (Aberdeen), is to have a member of the staff of the department (to which the students being taught belong) attending the lectures and commenting on the suitability and relevance of the material presented.

That we are really facing a serious problem here was pointed out by a number of correspondents. The stricter budgeting procedures which are an international trend, and the emphasis on cost per student put pressures on science and engineering departments (and others) to teach their own mathematics so as to reduce the "costs per student" in their departments. So, as well as concern for the level of mathematics, there is concern for the employment situation for mathematicians.

Once again I stress that the relative amount of space devoted in this report to service courses is inversely proportional to the amount of discussion which is necessary.

4 THE ROLE OF UNIVERSITY MATHEMATICS TEACHERS

In this final section we mention some trends concerning the role of university mathematics teachers. This role has changed considerably in recent years.

There is more personal contact with students than there used to be and time is devoted not only to their academic problems but also to their personal (and other) difficulties. Students expect more. Sometimes they even expect to understand without doing much work themselves, without reading around, etc. Despite the fact that in many universities the staff has grown more rapidly than the number of students, more time is spent helping students, correcting papers, etc. than before. Where the number of Ph.D. students has gone up, or where projects are used, large amounts of time are often devoted to a single student. As a result, there is a practically universal complaint about lack of time for research. In countries where there is a shortage of teachers, the teaching loads become extravagantly large. Luckily, this is often compensated by regular (sabbatical) leaves which ensure that the scientific level of the staff does not decrease. A second method which is used in several countries is to have two kinds of appointments, one regular and one "teaching only". This has the advantage that the non-creative mathematician who is a good teacher gets opportunities. The danger of being considered a second rate mathematician, or eventually actually becoming one, is non-trivial. A second problem is the decrease in the number of students, which has already started. Soon there will be members of staff for whom there is no work.

Luckily the trend of decreasing time available for research occurs simultaneously with an increasing emphasis on teaching skills at university level. This is reflected e.g. in the D.A. programme discussed in section 1.3.1. Another universal trend is an increasing amount of administrative work, committee work, etc. Some countries are suffering the difficulties of a parliamentary system in universities, which has made the process of administration and of taking decisions excessively slow. It is strange that so many complain bitterly about such waste of time and yet nobody actually does much to remedy the difficulties.

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PANEL

M.W. Al Dhahir (Kuwait), A.C. Bajpai (United Kingdom), H. Halberstam (United Kingdom), T. Hedberg (Sweden), J.M. Howie (United Kingdom), L. Iliev (Bulgaria), K. Kirchgässner (Federal Republic of Germany), A. Lichnerowicz (France), M.F. Newman (Australia), S.M. Sharfuddin (Bangladesh), G.S. Young (United States).

In addition to the panel members, D. Bushaw (United States) contributed to this chapter.

Chapter V

Adult and continuing education in mathematics

R. M. Pengelly

1 INTRODUCTION

1.1 The approach

At the 1976 Karlsruhe Congress on Mathematics Education, the theme Adult and Continuing Education in Mathematics emerged for the first time as an independent topic for discussion at an international congress on mathematics education. This development has been brought about as a result of the greatly increased interest in adult and continuing education in most countries in the world. The development of a discussion on this new topic presented a *major challenge* to the reporter, to the panel members and to all those who took part in the discussion. In a very real sense, those who involved themselves with this work were attempting to lay a foundation for the future discussion of this topic at subsequent conferences.

This chapter takes the broadest possible view of the subject area and atempts to show the way in which mathematics education fits into the field of adult and continuing education because:

- This is the first treatment of the topic, so it is important to set the work in its proper context.
- The developments in adult and continuing education which are currently going on must be understood if one is to see where mathematics education fits into the picture.
- The information available, on which to base the chapter, is inadequate to support a narrow mathematics education orientated view of the subject area.

This last point is an important one. While the literature describing the development of adult and continuing education is relatively poor, there are a number of reviews of the field to draw upon. On the other hand, the literature relating to the teaching of *mathematics* in adult and continuing education is almost totally non-existent. Hence, although this chapter provides information and observation on a range of issues, it is in most cases not possible to offer a critical analysis of the problems because of lack of adequate information. However, it is hoped that this chapter will serve as a challenge to all workers in the field to write up their experiences for the benefit of others.

Section 1 sets the scene by describing the theme in terms of: the area covered; goals and legislation; social and cultural influences; teacher/student relationships; and mathematics education within adult and continuing education. Section 2 aims at presenting a factual report on some current activities in the area covered by the report. The aim of section 3 is to give an indication of trends which are likely to influence the developments in this field. Section 4 deals with difficulties in identifying problems in distance study systems and with the improvement of such systems. Finally, in section 5, the challenge is emphasized and a proposal concerning the way to meet it is formulated.

1.2 Area covered

The range of activities currently being undertaken in the field of adult and continuing education in mathematics is substantial – varying from a brief series of broadcast talks to a full length degree programme – so that a completely comprehensive treatment of this theme is not practicable. Consequently, it has been essential to seek a basis for reducing the area covered by the chapter without destroying its utility. The selection of material for inclusion was influenced by two main themes which are of major importance in adult and continuing education, namely: *Mass Education for Mature Students* and *Distance Teaching*.

For a large number of areas it has only been possible to mention either significant or novel and interesting developments. The areas treated in this way are as follows:

- Education provided by commercial organizations for profit.
- Postgraduate or specialist courses.
- Courses with no formal class registration requirements and no formal assessment (especially broadcast courses).
- Pre-service teacher training courses (whether full-time or part-time) unless taught using distance teaching methods.
- Courses for training in vocational skills which have a mathematics component.
- Full-time or part-time courses which form part of a nationally recognized qualification, if they are taught in the classroom.
- In-service training courses for teachers, if they are taught in the classroom.

The existence of a largly free market in adult education in most countries has led to the creation of a variety of educational organizations serving the needs of adult students, ranging all the way from commercial profit-making organizations to what are essentially public utilities. Different organizations have been created to meet different goals and to serve different needs (see sections 1.3 and 1.4). Contrary to what happens in initial education, in the field of continuing education public service provision is relatively small and has come into the field relatively late. For example, Bouvier (1976) points out that in France the Ministry of Education is responsible for only 6 per cent of the students undergoing industrial training.

Correspondence Courses have always had an important role to play in adult education; the oldest Correspondence Colleges have been established for almost a hundred years. While correspondence education in many countries has developed as a result of commercial enterprise, it has also been used very successfully as a component in the public service provision of education. For example, Pentz (1975) reports that in the U.S.S.R. one third of all the students currently undergoing higher education are taking correspondence courses (a total of 1.6 million students in 1974). Also in developing countries, correspondence education has had a substantial impact. For example, a workshop on Correspondence Education in Africa held in Kenya in 1973, was attended by 43 participants from 16 different nations.

The last 10 years have seen a revolution in home study courses, in that traditional correspondence courses have been enhanced by the introduction of a variety of audio-visual teaching aids and the introduction of new forms of contact between teacher and student. This new form of home study course has been given the label "Distance Education". A report on Distance Education by the European Parliament's Committee on Cultural Affairs and Youth, the BACIE Report (August 1975), defines this type of education as having three characteristics: (1) spatial separation of student and teacher for all or almost all of the courses; (2) the use of teaching material in permanently recorded form, such as printed notes, films, cassettes, radio and television; (3) teacher control and goal directedness.

The same report concludes that an analysis in member countries reveals a fairly uniform pattern, namely that the *need* for distance education has arisen because conventional channels of education cannot satisfy the growing demand of the adult population for increased knowledge and instruction. A Council of Europe Report (1975) gives a review of the use of multi-media distance study systems at post-secondary level (in Europe, with a few examples drawn from the remainder of the world for comparison purposes). This report provides a clear description of the main characteristics of this new form of educational system and indicates how this approach has been put to use in a dozen different situations.

In almost all distance teaching projects mathematics is taught to a greater or lesser extent. Focusing on the distance teaching theme must not be allowed to distract us from the fact that a considerable amount of adult education (particularly vocational training) is still carried out in the classroom (often on a part-time basis). For example, in the United Kingdom, there were in 1972 about one million students undertaking vocational training courses on either a full or part-time basis in further education colleges. About 1 per cent of these students was undergoing some form of specialist training in mathematics (often at a relatively low level). Any report on adult education in mathematics must consider mathematics education in this sort of context. Since work of this sort lacks the "glamour" of distance teaching, information on the extent of activity in this area (even by number of students) is hard to come by. Every effort should be made to rectify this situation. In particular, effort should be made to compare and contrast success and failure at teaching mathematics – to groups of adults in the classroom and by distance study techniques to individual students – with a view to identifying the strengths and weaknesses of the two approaches (and perhaps comparing the cost).

1.3 Goals and legislation

In order to understand a number of the developments discussed elsewhere in this chapter, it is necessary to consider the goals for adult and continuing education programmes and the way in which government legislation can influence such programmes. The goals of mathematics education itself form the subject of Chapter IX.

It is possible to identify four primary goals for adult education which appear to apply both in developed and in developing nations. These are (Farnes, 1976):

Work-orientated education – the goal being to help the individual to work more effectively.

Family and community education – the goal being to create a happier and healthier society.

Professional and para-professional training – the goal being to increase the specialist skills available in society.

General or basic education – the goal being to raise the level of literacy, numeracy and so on in society, thus enabling individuals to participate more fully in the development of society.

Hlavaty (1976) points out that there ought to be added to this list a fifth general goal which stresses the needs of the individual as an individual – requesting an education to develop the fullest potentialities of each individual student and to give him a sense of fulfilment. (See also Melo, 1976.)

These general goals are complemented by more specific goals related closely either to the student's or society's needs. For example, the BACIE Report (August 1975) identifies the desire to "get on in life" as a powerful motivation for students seeking adult education. In addition, adult education can become an integral part of government policy. For example, in a report on vocational training in the European Economic Community (BACIE, June 1975) it is stated that: "The maintenance of full employment is a cornerstone of the community's economic as well as social policy. Attention should therefore be paid to education to ensure that the adaption of the labour force keeps pace with the changing demands of the E.E.C's economy." And further: "The concept of training as a continuous process of education at all stages of a worker's life, and at all occupational levels, is accepted as a key factor in safeguarding the essential mobility of the labour force."

A report published in the United States by the National Advisory Council on Adult Education (1974) gives an even broader statement of policy: "Adult education should as a matter of deliberate design seek out the needs for its services generated by the growing number of transitions that are being made among schooling, work and retirement, and mount programs which will help those making these changes continue to be as active economically, vocationally or in leisure pursuits as they need in the changing patterns of their lives."

Because of this close relationship between adult and continuing education and government policy, it is natural that nations have introduced legislation which seeks to control and influence the development of adult education: for example, Bouvier (1976) describes the law regarding adult education in France.

It is not only in developed countries that we find substantial government influence in the development of adult education. For example, Ahmed et al. (1975) describes government involvement both in the development of basic general education programmes and in the development of employment related education in a number of developing countries. The wealth of goals in adult education in developing countries is evident from the Commonwealth Secretariat Report on Correspondence Education (1974).

Another goal of adult education is the creation of greater level of "social equality" (CERI/OECD Report, 1975). The Council of Europe Report (1975) ascribes much of the impetus behind the massive expansion of adult education, especially in the post-secondary area, to the desire to break the "exclusiveness" in the restricted provision of secondary education.

A rather different view of determining the goals and direction of adult education is given by Holmberg (1975) who argues that existing syllabuses, curricula and examinations, together with an analysis of the wishes of the "consumers" (both the student and his potential employer) and an analysis of the opinion of practitioners (those who possess the competence which the study is intended to provide), should be the driving force determining goals in adult education.

At present, there is no information available about how the goals and legislation which affect the development of adult education will influence the teaching of mathematics in adult education. It would be very useful to try and obtain some sort of picture of the changes that are taking place and, in particular, to discover whether the same sort of developments are occurring both in developed and developing nations. It would also be interesting to know to what extent the goals in adult education are leading to a demand for general cultural courses about mathematics rather than courses that aim to teach specific mathematical skills and concepts.

1.4 Social and cultural influences

A good example of how social pressures can influence adult education is given in Bouvier (1976) where the influence of a professional agreement between unions and employees in France is described. This agreement led to a change in attitude towards adult education so that in the period 1972 to 1974 there was a 60 per cent growth in the number of students taking an adult education course (in fact, about one employee in three now takes some form of adult education course). Bouvier also explains how education programmes are influenced by social and cultural attitudes. For example, if an adult's commitment to education is seen as an investment of time and money towards meeting a particular need, then the student expects that the objectives of the education will take into account his educational needs. Thus the richness of the field of continuing education stems from the variety of human situations with which it is intended to cope.

Another major influence to be taken into account is the impact of admissions policies and certification. A viable continuing education programme, especially if viewed as providing "recurrent education", requires a *radical review of the criteria for admission to post-secondary school education* (CERI/OECD Report, 1975). In most developed nations, a gradual acceptance is emerging that "experience" is likely to be the thing that most helps people to take advantage of education and in this context *work experience* may be seen as a valuable admissions qualification. Cultural and social changes – which lead to the adoption of such radically different admissions policies for post-secondary education – must in turn have an impact on the courses themselves. This will be particularly true in the case of a subject like mathematics, where it is not possible to replace basic mathematical skills with some "general work experience" and continue to teach the course in the same way. At present, there appears to be no information on which to assess the impact of this change on the teaching of mathematics. However, the Open University (Mathematics: A Foundation Course) which operates an open admissions policy, has shown that courses can be designed to *successfully teach post-secondary school mathematics* to students with a limited previous experience of mathematics.

As far as *developing countries* are concerned, the social and cultural influences on the determination of "what courses to offer" are described by Perraton (1976) in relation to the situation in Botswana. From the material given in that paper it is clear that the social and cultural pressures in developing countries will lead to most of the mathematics education being carried out as part of a formal education programme.

1.5 Student and tutor roles and relationships

Student and tutor roles and relationships in adult education are different from those found in normal initial education. The introduction of distance teaching techniques is the factor which appears to have had the most impact on the teacher and student roles and relationships. Perraton (1976) explores the role of teachers and students and their relationship in distance learning situations. Two main points emerge.

The first point is that distance learning is an unusual human activity, and for that reason it is difficult. The argument in support of this point is basically that learning is normally a social activity. If this view is accepted, then distance students are in an unusual learning situation. This means that there are no precedents for developing the roles and relationships of teachers and students in distance learning.

The second key point is that there is a need to introduce an *element of criticism* into a system which relies exclusively on pre-prepared materials. This means that an important aspect of the students' relationship with a tutor stems from the teacher's need to obtain feedback in order to improve the teaching materials. This gives rise to a situation in which tutors are learning from their students as well as teaching them. This point is developed by Harrison (1974) who argues that independent, self-directed critical thinking is a precious quality in both tutors and their students and is to be encouraged at all times. So the teachers' roles in a distance study system should be to encourage a sturdy and competent independence in their students' thoughts and outlook. Sida (1976) has pointed out that when discussing mathematics education and when creating courses and developing teaching methods, no mention is ever made as to how one teaches independence in mathematics courses.

A development in students' relationships with a tutor which has been applied in teaching mathematics is described by Daniel et al. (1975) and involves the tutor organizing group work. The author reports that those tutors who put themselves in the role of instructors fail to create an adequate working relationship with their group. The best tutors are those who become proficient at using the resources of the group itself to solve problems.

The real impact of multi-media distance teaching systems on student and tutor relationships is emphasized by Adler (1975) (which describes the experiences of Turret Correspondence College, South Africa) and in which the author points out that in multi-media education the teacher is no longer the main resource of the student, but simply one amongst many resources. In this situation, "teaching students" is only one of his duties; he has also to cope with generating group work and discussion, helping students individually and in groups to use educational resources at their disposal, and perhaps most important of all (especially in mathematics), helping students individually with their difficulties and counselling them on their path through the course. The author points out that while this enlarged role for the tutor has been recognized by the creators of the educational system, it is not always recognized by the students themselves, who, because of previous experience (especially experience arising from other forms of adult education), often expect and even demand "normal lessons". This view is reinforced in a conference report published in the United Kingdom by the National Council for Educational Technology (1976).

The Council of Europe Report (1975) highlights a different aspect of the teacher's role in distance education, the fact that it brings into existence new relations between the teachers themselves. In order to create multi-media courses, it is necessary to use teams of teachers to create the educational material. The advent of "team teaching" brings to education a new opportunity for critical appraisal and evaluation of the whole teaching and learning process. Mason (1976) gives some insight into what happens in team teaching situations (the author describes "life" inside three mathematics course teams at the Open University).

Bouvier (1976) points out that since an adult's commitment to education is often an investment designed to meet a particular need, there exists in some sense "a contract" between the teacher and the student which requires that the teacher enables the student to measure his progress towards his goal, so that one of the tutor's important roles is to help the student assess how he is getting on. Such "contractual relationship" between the teacher and the taught, leads to the students being a good deal more *critical* of the courses presented to them. They are quite likely to condemn course materials which appear difficult or artificial and unrelated to their present needs as they see them (this sort of attitude can lead to particular difficulties when mathematics is taught as a specialist subject). Bouvier (1975) concludes that despite the fact that changes in the tutor's role put greater burden on the teacher, it also gives them much greater satisfaction and, more important, greater opportunity of being educated themselves during their teaching process.

1.6 Mathematics education within adult education

The factors which influence the scope and content of mathematics programmes in adult education are directly related to the general factors influencing adult education described previously; (i) meeting the students' needs; (ii) meeting society's needs; (iii) curricula and certification. It is interesting to consider these factors by examining how mathematics education fits into an adult education programme. In this connection, Sida (1976) points out that the least one can do in adult education is to show the connection between mathematics and the real world in which we live. This requires that particular attention is paid to the clientele and their special needs.

In correspondence education offered by universities in the U.S.S.R (Pentz, 1975), the mathematics contained in the courses provides a service function so that mathematical knowledge and skills are taught in order to enable the student to study the other components of his course. It is the course as a whole that has been designed to meet society's needs and it is the demands of the course as a whole that have been used to determine the content of the mathematical components in the courses. This approach, which must be common to many other situations, can lead to the courses containing a substantial amount of mathematics. For example, a table given by Pentz (1975) shows that in a degree programme in machine technology, the student spends about a quarter of his time studying mathematics, while in a degree programme in chemical technology he spends about a fifth of his time studying mathematics.

Bouvier (1976) points out that much of the mathematics that is taught is closely integrated into the teaching of other subject material. A common approach is to involve the student with problems which relate to his own environment. By showing the student how to solve several problems, one can build up an understanding that methods and procedures are more important than results themselves. This leads naturally to the concept of a mathematical model and also to the discovery that certain mathematical ideas, common to a variety of problems, are worthy of study in their own right. In this approach mathematics is taught as a tool for helping to solve problems, the subject and content of the mathematics elements in the course being determined by "needs" identified as a result of the student's attempt to tackle a range of problems. Sida (1976) points out that service teaching of mathematics is becoming increasingly important and that this brings with it the challenge to make relatively sophisticated ideas palatable to a broad cross section of society. This view is reinforced by the adult higher education programme offered by the *Conservatoire National des Arts et Métiers* in France in which many of the mathematics programmes are explicitly linked with the needs of other sciences and technologies (*Enseignements 1976/77*).

The most natural way in which mathematics appears as a discipline in its own right in adult education is in those situations in which a mathematics course (of some appropriate level) is taken in order to achieve a recognizable qualification (usually based on some national system of certification). For example, Heidt (1975) describes a situation in which examination certificates issued by a "Radio College" were accepted either as part of the qualification for university admission or as an additional component in a teaching certificate.

Specially prepared mathematics courses also appear in the degree and certificate programmes offered by a number of distance teaching institutions. For example, the German Institute for Distance Studies described by Heidt (1975) offers courses in mathematics for secondary school teachers as part of their programme. The Open University (*Courses Handbook 1976*) offers a variety of courses in mathematics as components which a student can choose to take as part of his degree programme. A number of the other distance teaching institutions described in the Council of Europe Report (1975) offer mathematics courses as part of their programme, while the Canadian system described by Daniel et al. (1975) offers a specialized degree in mathematics.

One development in adult education in mathematics which does not appear to have been pioneered yet, is the use of distance teaching techniques (particularly the mass media of radio and television) to teach numeracy on a large scale. Radcliffe (1976) describes how a series of radio and television broadcasts have been used to mount a successful attack on the literacy problem – can the same approach be applied to numeracy?

Another interesting question arises from considering adult education in developing countries. Small (1976) describes the current continuing education programmes in Zambia and stresses the importance of non-formal, non-credit education in bringing swift results for those in employment (what they learn to-day they can apply tomorrow). Where, if anywhere, does mathematics education fit into this situation?

2 WHAT IS HAPPENING?

The aim of this section is to present a factual report on some current activities in the areas covered by the report. Unfortunately, the amount of factual information available on the content and style of mathematics programme in adult education is currently very small – there is not even much statistical information on student demand and student performance available. It is one purpose of this chapter to stimulate the publication of information which will help fill this gap.

The courses offered in adult education often do not follow the division between primary school, secondary school and higher education that is exhibited in initial education. However, dividing up the activities in programmes at elementary level (section 2.1) and at advanced level (section 2.2) helps to bring out the relationship between the situation in adult education and the situations described in Chapters I to IV.

Sida (1976) points out an important characteristic feature of adult education in mathematics – namely the fact that one is often faced with a student who has at some time failed to grasp, or complete, a mathematics course and probably tends to regard that as a failure. The challenge then is to find new ways of presenting the same sort of material; to make what was dull exciting, what was obscure clear, what was pointless relevant and what was learned by rote understood.

2.1 Programmes at elementary level

We are here concerned with the mathematics programmes which are normally taught in primary and secondary schools – although the approach and presentation is very different. In adult and continuing education programmes, mathematics at this level is usually taught in an attempt either to provide the student with basic mathematical skills which he can apply later in his education, or to provide skills he can apply directly in his trade or profession. The material below describes the situation in the United Kingdom which is reasonably typical of the situation in other developed nations.

The simplest mathematics courses are those which are taught as an integral part of some other adult education programme. Houlton (1973) describes a course in "relevant mathematics" which formed part of an industrial studies course for trade unionists presented by the Extension Studies Department of the University of Liverpool. The mathematics presented in the course comprises an exploration of certain basic ideas set into a *relevant industrial context*. For example, the idea of an arithmetic mean is explored by investigating the meaning of the phrase "average wage". The material not only describes how to calculate an average, it also explains how averages are used and interpreted in practical situations (for example, wage bargaining). The emphasis is very much on teaching mathematical ideas and techniques which the student can put immediately to work in his other studies and in his working life.

The National Extension College (*Guide to Courses 1975/76*), which is a non-profit making correspondence college run by an Educational Trust, offers students a whole range of mathematics courses (sixteen in all) ranging from a simple first course in mathematics (which assumes only the ability to do arithmetic with whole numbers) to courses which are prerequisite to normal university entrance. Such a programme provides students with an opportunity to acquire credit certificates for those parts of a formal mathematics education which they failed to gain during their time at school. The College also offers one or two special refresher courses; in particular, a preparatory course for admission to the Open University Mathematics Foundation Course.

Further education colleges, local technical colleges and evening institutes throughout the United Kingdom offer students a similar range of courses, only in these cases the medium is classroom teaching on either a full or part-time basis. These colleges also offer a range of other mathematical courses for nationally recognized qualifications (for example, National Certificate and National Diploma courses) which form part of many vocational training programmes.

There is also a considerable number of courses in mathematics being taught on a non-formal, non-examination basis. These courses are designed to serve a variety of needs and are taught by several different agencies (almost exclusively by classroom teaching either on a part-time or full-time intensive basis). A typical example of this sort of course is "An Introduction to Modern Mathematics" aimed at parents of children in primary and secondary schools who are interested in following the progress of their children's education and who want to know what is happening in mathematics education. There are other courses, of a more general cultural nature, which are simply "about mathematics" and there are yet other courses which teach the application of mathematics to current social problems (for example, metrication and value added tax). All these courses are aimed at adults who are interested in acquiring information, insight and some understanding of the role of mathematics in some particular situation. No information is available to indicate either the breadth and depth of courses which are offered, or the characteristic and number of students involved. Unfortunately, there appears to exist no published information about the factors which influence mathematics education programmes at this level. For example no information seems to be available on: teaching styles; use of media; teacher and student relationships; interaction between mathematics and other subjects in the programme. Therefore, no comparison can be made directly with the situation as reported in Chapters I and II.

2.2 Programmes at advanced level

This section covers projects involving the teaching of mathematics at post-secondary level (college and first degree level but not specialist or postgraduate courses). As with the programmes at elementary level it has proved difficult to find material which gives a reasonable picture of the extent of activities in this area.

In the United Kingdom, the situation is reasonably straightforward. On the one hand, the Open University (*Courses Handbook 1976*) offers post-secondary adult education at degree level in a wide range of subjects including mathematics. The Open University degree is credit based and the student is completely free to follow a programme of his own choosing. Consequently, the mathematics courses offered by the University are offered as a contribution to a general degree programme rather than as a basis for a specialized degree in mathematics. The topics covered resemble those normally found at first and second year level in United Kingdom universities. The Open University is a multi-media distance teaching system and students follow the courses largely by private, independent study. Limited facilities are available for face-to-face contact with a tutor and some courses include a one week residential summer school.

The only other provision at post-secondary level in the United Kingdom is the provision at some universities, polytechnics and further education colleges of part-time degree programmes. These degree courses are normally taught by part-time day release and evening classes. The students follow similar, if not identical, programmes of study to those taken by the full time undergraduate students. Admission to most of these part-time degree programmes is dependent upon obtaining a pass in a recognized public examination (this is in contrast to the Open University mentioned above which operates an "open admissions policy" based on no formal educational requirements). These part-time degree programmes offer students the opportunity to follow either specialist degree courses in mathematics or courses in which an element of mathematics is combined with one or two other subjects.

The situation in the remainder of Western Europe and the United States appears to be rather similar; for example the *Conservatoire National des Arts et Métiers* in France offers a similar programme (*Guide de l'élève 1976/77*) and so does the International Centre for Degree Studies in the United States (*I.C.S. Catalog 1976/77*).

A glimpse of the situation in developing countries can be found in the information given by Mackenzie et al. (1976) in a report which describes 17 case studies in distance teaching at post-secondary level in 13 different countries of the world. Unfortunately, this report does not devote much space to the subject matter of the teaching programmes. However, it is clear from the report that one developing country (Iran) is planning on providing degree level courses in mathematics aimed at the pre-service and in-service training of teachers. Similar ventures are believed to be under way in Mexico and a number of other Latin-American countries.

An interesting development at post-secondary level is the use by one institution of mathematics education, courses produced by another independent distance teaching institution. This has been successful in at least two situations in which other institutions have used courses produced by the Open University. Thus, the University of South Africa has made use of the introductory mathematics course and two computer studies courses and the University of Maryland, United States, has used the introductory mathematics course produced by the Open University as a component in its adult education programme. The success of these two projects underlines a claim advanced in the Council of Europe Report that distance study systems in whatever country they may be set up, and irrespective of the level of education involved, represent a contribution to the constitution of a "knowledge bank" on which all can draw. If this claim is sustained the impact on the teaching of mathematics at post-secondary level could be substantial because specialized mathematics courses created in one institution could be adapted and used by another. This would not only save manpower, but would serve as a basis for transferring pedagogical innovation from one institution to another (and even from one country to another).

A development in continuing education in mathematics, currently flourishing in the United States, which does not appear to be matched by similar level of activity in Western Europe, is the "extension short courses" offered by many of the large universities, for example, by the

University of California in Los Angeles (UCLA Extension, *Continuing Education in Engineering and Mathematics,* 1975). These courses are of degree standard or beyond and are usually offered on a full-time intensive basis over a period of between three days and four weeks. The courses are classroom taught although many involve the use of additional audio-visual aids to support the lecturer. Many of the universities involved will contract to provide the same or similar short courses for a particular company or government organization on the organization's own premises and specifically modified for that organization. According to the published literature, the short course programme is designed to provide participants with the latest both in theory and application by creating a close tie between university teachers and professionals in industry and government. The mathematics covered by such programmes is largely concerned with modern mathematical techniques used in management and research.

3 NOVEL FEATURES AND TRENDS

The aim of this section is to review some of the novel features of existing projects and to give an indication of current trends which are likely to influence developments in the areas covered by this report. Most of the information currently available is of a general nature; little has been written which is specific to the teaching and learning of mathematics.

Before looking at some more special tendencies, it is worth identifying three *general* trends which seem to apply to the development of adult and continuing education throughout the world. First, there is the trend towards *expanding the range and scope* of the programmes provided to meet the needs of all interested adults. This trend is clearly illustrated, for example, by the current developments in the Federal Republic of Germany (Heidt, 1975).

The second trend is towards the production of courses more directly *related to specific occupational needs* which guarantee a specific level of education and the provision of a nationally recognized qualification. This trend towards courses directed at a particular clientele is illustrated by the developments both in Canada and the Federal Republic of Germany (Daniel et al. 1975, and Heidt, 1975).

The third trend, and perhaps the most important of all, is one of very rapid *growth in the number of students* taking part in some sort of continuing education activity. This trend is illustrated by the experience in France (Bouvier, 1976).

The situation in Canada and the Federal Republic of Germany shows that the second of these general trends applies equally to mathematics education. Currently, there is no definitive evidence to show whether the other two trends also apply to the development of the teaching of mathematics in adult and continuing education.

3.1 Use of educational media and resources

This section reviews the use of educational media and resources in adult and continuing education and tries to indicate how these developments have influenced (or might influence) the teaching of mathematics. Unfortunately, little has so far been written about how the use of educational media and resources in adult education has influenced the teaching of mathematics, so the position has to be reviewed on a rather general level. Further, there appears to be little information available on how new educational media have influenced the development of classroom teaching in adult education. Consequently in this and the subsequent section, the discussion is almost entirely concerned with distance teaching systems. For the reasons given in sections 1.5 and 1.6 above, it is natural to expect that many of these developments will eventually influence classroom teaching in adult and continuing education.

The most important trend in the use of educational media and resources has been the development of multi-media distance teaching systems in which the use of traditional correspondence material is supported and supplemented by a wide variety of audio-visual aids (slides, tapes, cassettes, records, and so on) by open-circuit radio and television broadcasting and by the introduction of limited forms of personal contact between the student and the tutor. (The BACIE Report, 1975, traces these trends.) In the analysis given in the Council of Europe Report (1975), it is pointed out that many of the originators of distance study systems who start out with the intention of using a single preferred medium, gradually evolve towards a multi-media system so that, for example, a course which started out as a radio course may end up as a multi-media course, the radio broadcasts being supplemented by correspondence materials involving homework to be sent in and marked by a tutor. We appear to be approaching a situation in which it is most unusual for only one kind of medium to be used.

Teaching by correspondence plays a dominant role in most distance teaching systems. The most significant development in this area has been the use of course teams (by the Open University and others) to produce high quality correspondence materials carefully matched to the needs of the independent learner. This development appears to be an important one in the field of mathematics education since the use of such materials gives rise to student pass rates superior to those previously obtained by using traditional correspondence materials (simple instructional material, standard text books and test papers).

Another practice which is now so widespread that it deserves examination is that of bringing students together into a group with either a tutor or a counsellor present. In most systems, students meet in regional or local groups with a tutor who can help them to assimilate and understand the material being taught. In some cases, for example in the U.S.S.R. (Pentz, 1975), these groups are used to introduce a class teaching element into the course and form a significant part of the student's study time (about 30 per cent for the courses described). However, in many other situations in both developed and developing nations (see Perraton, 1976; Daniel et al., 1975; Adler, 1975), the value of the study group is seen as lying in the opportunity it provides for the tutor to help the student become an independent learner, rather than in the opportunity it provides to offer classroom tuition. The Open University in the United Kingdom tries to "get the best of both worlds" by leaving the tutor and his own group of students to decide what to do with the face-to-face teaching sessions. The Open University has also introduced into a number of its courses, a much larger "classroom teaching element" by requiring that the students attend a one week full time intensive summer school (held in conventional universities during the summer holiday period). The summer schools are used to provide a wide variety of educational activities which cannot be provided in any other way; for example, laboratory work in science and in mathematics active problem solving sessions. So far, nobody seems to have evaluated the significance of face-to-face tuition as a component in teaching mathematics in a distance study system. However, it seems self-evident that such group contact is valuable to a student who is taking a mathematics course because it helps him to develop relationships between himself, his tutor and his fellow students, which enable him to not only tackle some of the difficulties he is having, but also to make a more realistic assessment of his own progress.

Broadcast media – radio and television – form an integral part of a great many distance study systems. However, according to the Council of Europe Report (1975), they remain the media which gives the most trouble, because the production of lively, high-quality broadcast teaching involves bringing together teachers, producers and a variety of other technicians in an environment which encourages creative and constructive work. The best of the Open University teaching materials shows what can be achieved in this area. As far as the teaching of mathematics is concerned, experience at the Open University suggests that the effective use of broadcast media requires the investment of a great deal of time, care and attention. Given sufficient effort on the production of teaching materials, the broadcast media appear to form a valuable component in teaching mathematics at a distance.

The scale of the resources and the extent of the academic knowledge and experience needed to produce successful distance study courses in higher education call for and stimulate many forms of co-operation between conventional institutions of higher education. For example, it is reported (Council of Europe Report, 1975) that in France the universities are combining their endeavours, through inter-university centres for distance study, and there are moves towards the establishment of Regional Federations of Universities. Similar developments are taking place in the Federal Republic of Germany (Heidt, 1975). The nature of co-operation between institutes can vary. For example, in the U.S.S.R. (Pentz, 1975) the conventional colleges co-operate with the correspondence colleges by creating and manning the local study centres. A similar thing happens in the United Kingdom in relation to the Open University only on a less formal basis. In Canada (Daniel et al., 1975), the universities have actually co-operated in the production of course materials. Experience at the Open University suggests that all these forms of co-operation are of mutual benefit to all concerned when it comes to the teaching of mathematics.

One medium which is coming into increasing use over the last few years is "teaching by telephone" (see, for example, Short, 1974). A number of correspondence colleges (for example, the National Extension College in the United Kingdom) provide for telephone contact either between the student and his tutor or between the student and the central system producing and distributing the teaching materials. The telephone is used primarily to allow the student to ask questions and for a tutor or counsellor to respond. At the Open University and elsewhere, experiments have been going on with more ambitious forms of telephone tuition, which involve an attempt to provide "telephone tutorials" as a substitute for face-to-face tutorials. The experience gained so far is very limited. In particular, it is not yet clear how well one can use a medium that has no visual channel in the teaching of mathematics.

Sida (1976) points out that a peculiarity of learning mathematics is the fact that a student can be brought to a standstill over a very small point, unable to progress until helped out. Since it is probably impossible to avoid such obstacles, the challenge is to find a solution – travelling to a three-hour tutorial is hardly the appropriate solution to a problem which one minute's conversation would solve.

One new medium which certainly has an impact on the teaching of mathematics is the hand-held calculator. The provision of a calculator has two advantages (Bouvier, 1976). First, it enables students to tackle realistic examples (often examples associated with their own work) and second, it allows students to explore – and possibly even to discover – important mathematical ideas (the concept of a limit, for example). Experience at the Open University suggests that there is a long way to go to discover how to use this new media effectively. However, there seems no doubt that this one device may have a bigger impact on the teaching of mathematics than any of the other new media which has come into use over the last few years. (See Chapter XIII.)

No review of the use of media in teaching, especially in the teaching of mathematics, will be complete without mentioning the computer. In the teaching of mathematics generally, the computer possesses many of the same advantages as the pocket calculator (while it is more difficult to use, it provides a great deal more scope). However, in distance teaching systems it is currently difficult to take advantage of this inherent power because there is currently no convenient and cost-effective way of making computing facilities available to students in their own homes. However, the computer remains an important component in many distance teaching systems because it can be used both to "manage the system" and to monitor and record the student's progress. Knowledge testing through the use of computer-based questionnaires, if sensibly applied in conjunction with other evaluation methods, can make a teacher's job easier by giving him a clearer picture of the student's knowledge throughout the course. This information can be used by the student himself to help him judge his progress and by the teacher in his assessment of the student. As far as the teaching of mathematics is concerned, the use of computer-marked tests presents many difficulties (see, for example, Pengelly, 1973). However, it seems likely that it is only by the development of more sophisticated computer-based knowledge testing systems that we will ever find a cost-effective way of improving the feedback element in distance teaching systems which is so important in the teaching of mathematics.

3.2 The design of learning systems

The use of educational media and resources reviewed in the previous section can be directly related to present and future approaches to the design of learning systems. Clearly the primary goal of the designers of a learning system is the creation of an economical and effective system. The challenge facing the designer of a distance teaching system is to discover ways and means of using the wide variety of media open to him (see section 3.1) to their best advantage.

A distance teaching system requires a great deal of time and effort invested in the design of the learning system itself if the system is to meet the needs of the independent learner. For example, great care is needed in matching the educational media used to the student's environment. Unfortunately, as Mackenzie et al. (1976) point out in their review of open learning systems, designers are involved with devising systems based upon technologies which they are still learning how to use. In the case of the planning of the Free University of Iran, an attempt was made to find a set of practical procedures, based upon sound psychological and pedagogical foundations, which would allow planners to define overall objectives for the different media which would serve as guidelines for the selection of media between and within individual courses. It was found, however, after considerable discussion, that no significant criteria could be elucidated, and a series of difficult analyses were often found to end in "common-sense decisions".

The aspect of the design of a continuing education programme which requires most attention from the designers is that of "coping with diversity". Diversity arises in two ways, which shall now be described.

First, the course which is being designed may be intended to serve a variety of different, often quite distinct, purposes. For example, in the Open University, the introductory mathematics course is supposed (at one and the same time) to meet the needs of both students who simply want an introduction to what mathematics is and what mathematics does, and the needs of students who are going to go on and study a variety of other courses with a significant mathematical content (some of them mathematics courses, others courses in physics, electronics, and so on). This diversity of purpose makes it difficult to define the exit behaviour for the course in terms of a sensible collection of behavioural objectives and in mathematics leads to an obvious conflict between providing a broad introduction and the detailed preparation for future study.

Next, diversity also arises from the student's background. The student groups taking a particular course are very heterogeneous (see for example the Council of Europe Report (1975) which gives the composition of the student groups for a number of adult education courses). The heterogeneity of the student population gives rise to two problems. First, it means that the students possess widely differing educational experience, so that, for example, the Open University is faced with teaching the same mathematics course to students with a range of experience of mathematics which extends from little more than primary school mathematics all the way to formal university entrance requirements and beyond. An institution can, of course, remove this difficulty by adopting some form of entry qualification for the courses. However, this tends to simply move the problem from one place to another, because it will lead to demand for adult education programmes designed to enable students to attain the entry qualifications. Next, the heterogeneity of the student body presents the designer with a problem which stems from the students' different interests and motivations. This means, for example, that when teaching a subject like mathematics, it is hard to know how to select examples to illustrate the ideas in the course, because an example drawn from one area and acceptable to one group of the student population may be much less acceptable to another group and may therefore reduce the latter group's motivation. This makes the teaching of "applied mathematics" in adult education a challenging task. The real challenge is to find a way of putting across valuable and interesting examples without too much technical detail. This task is made even more challenging if, as is suggested by Bouvier (1976), the aim is to match the course to the student's real needs and choose mathematical examples and problems of direct concern to him. One approach is to supplement a core of common content with a range of examples adjusted to the particular needs and requirements of a certain group of students.

A second problem for the designer of distance study systems is coping with a relatively small number of students spread over a geographically large area. Experience in the Open University suggests that designing a distance study system which can actually help students become effective independent learners is a difficult task. This task is particularly difficult in mathematics where even the smallest of difficulty (for example, failing to grasp the meaning of a single symbol) can lead to the student being unable to complete the study of an item of work. In order to cope with the needs of remote students, there is need to do much more research into the way in which mathematics is learned and in particular into the way in which the students use the teaching materials provided.

A third important problem facing the designer of a learning system is the method of assessing the students' attainment and of awarding course certificates. The method by which students' attainments are assessed and course certificates issued can often colour the whole attitude of both the teachers and the students. Since (as pointed out in the Council of Europe Report, 1975) the methods used for assessment are often the same as those used in traditional education systems, it is hardly surprising that many of the student and teacher attitudes that grow up are those which are found in normal education. Many institutions (see for example the Council of Europe Report, 1975) insist that the examinations taken by the students in adult and continuing education should be the same as those taken by students regularly attending courses in other educational institutes. This approach is defended on the grounds that it is necessary in order to achieve "credibility". However, there are equally strong arguments to suggest that a new kind of teaching calls at least for some new type of examination (if not the abolition of the examination itself). This problem affects those designing mathematics courses because the design of a successful mathematics examination depends critically on matching the examination to the students' knowledge, experience and attitudes and this is very difficult to do in a distance study environment.

Another aspect of assessment which requires very careful attention from the designer is the continuous assessment component of the course. Providing some form of continuous assessment is absolutely essential in a distance teaching system because both teacher and student need some way of monitoring performance. Students can of course be provided with "various forms of self-marked tests". However, such tests do not seem to be very satisfactory used as a component in a distance teaching system because a student working in isolation has no way of judging the meaning of the result of his test, he cannot even compare his performance with that of fellow students. Some of the difficulties involved in creating the continuous assessment component for Open University mathematics courses are described by Pengelly (1973).

Experience at the Open University suggests that the use of team teaching techniques is one way to tackle the design problems mentioned in this section. To be successful, the course team should be created in such a way that it can draw upon the experience of those who have tackled and successfully solved some of the design problems as part of their previous work.

4 PROBLEMS AND THE SEARCH FOR SOLUTIONS

Material so far published is inadequate as a basis for making a critical analysis of the problems that face adult and continuing education programmes. Mackenzie et al. (1976) are the only authors who have attempted to set down authoritative statements about recent developments in a wide variety of context and to make a comparative analysis of some of the problems facing distance study systems used in adult and continuing education. No similar analysis appears to exist for adult education programmes taught in the classroom. More significant from our point of view, no comparative analysis has been made of the teaching of mathematics or any particular subject in adult and continuing education.

In what follows, I have concentrated on distance study systems as this, in my view, is the area where it is possible to make a reasonable estimate of the main problems in teaching mathematics to adults. More important, in this environment it is possible to systematically devise and test improved teaching systems so that action can be taken to solve any problems that are identified. It seems reasonable to hope that the major problems involved in teaching mathematics to adults are the same no matter what the audience is or what teaching system is used. If this is the case, then teaching problems identified in teaching mathematics using distance learning systems and any solutions which are found to these problems should in large part carry over to the classroom teaching of mathematics to adults.

4.1 Identifying problems

A major difficulty with any system of education is actually identifying the problems that exist in the system. For reasons explained earlier in the chapter (section 1.5) those concerned with the operation of distance learning systems are committed to making a systematic analysis of the effectiveness of the teaching systems as their only means of improving the quality of the education system. This point of view is clearly explained by Bates (1974) who discusses the Open University's experience of using part-time tutors to help evaluate the teaching system. In the same paper, the author makes the very important point that if we are to identify problems, then feed-back must be collected in a systematic and a reliable manner. Failure to meet this constraint will lead to an imprecise identification of the problems in the system and this in turn may lead to unnecessary and even ill-conceived alterations to course materials.

Experience at the Open University suggests that in fact it is much easier to specify the need for feed-back than to find effective and reliable ways of obtaining it. One of the basic questions (Bates, 1974) is where the feed-back comes from. It might be thought that the appropriate source of feed-back on a course is the students themselves. Experience suggests that this is too narrow a view, for while student feed-back is essential, it has severe limitations as a sole source of feed-back. The main problem is that students in general are unable to suggest how improvements to the system might be made. This difficulty is due to two reasons: first, the students do not know enough about the subject matter or about teaching methods to know how else the subject might be taught. Second, if they are having difficulties it may be impossible for them to identify and explain the cause of their difficulties. For this reason, any systematic feed-back system must also seek feed-back from the teacher (course tutors in the Open University case) since the teacher is capable of distinguishing between weaknesses due to the course material and weaknesses due to students' shortcoming. The tutor is also capable of bringing forward suggestions about how improvements might be made. Certainly in the case of mathematics teaching it is very important to obtain feed-back from tutors because students are often unable to explain the difficulties they encounter in learning mathematics. In fact, Open University experience suggests that many of the difficulties students experience in learning mathematics are due not to the inherent difficulty of the subject material or even to an inappropriate teaching style or approach but to simply an unfortunate choice of phrase or a failure to adequately explain some relatively minor step in an argument.

Another valuable source of feed-back in a distance learning system is the assessment mechanism. The Canadian experience (Daniel et al., 1975) suggests that there is a need to develop assessment systems which are properly adapted to teaching at a distance. They argue that this is important not only in order to provide adequate feed-back for course designers and implementors but also in order to "be helpful" to the student. It is important that the assessment system encourages two-way communication and involves not only those centrally responsible for the production of the course but also those teachers (usually part-time) who are responsible for helping the students learn from the materials provided. Certainly the continuous assessment and the examination have to serve a broader purpose than simply providing a basis for grading students.

The authors of the Council of Europe Report (1975) bring out another very important point about the use of examination and assessment feed-back (essentially student performance data) to estimate the effectiveness of a teaching system. They argue that student grades should not be used on their own as a measure of the effectiveness of a teaching system, and point out that it is necessary to have adequate population statistics (on level of education, aspirations and so on) and other information on student achievement of his own goals (how well his needs are met by the course he is taking) before one can make a proper assessment of the effectiveness of a distance teaching system. This broad view of assessing the effectiveness of the system is particularly important when it comes to the teaching of mathematics, because it is inevitable when mathematics is taught to adults with limited previous experience of mathematics that the results achieved when measured directly by student performance data are significantly worse than results achieved by students following courses in the humanities and the social sciences. However, this observation does not excuse mathematics education from collecting and systematically analysing student performance data. Burt (1976) describes work done at the Open University in connection with the evaluation of mathematics courses and the attempts which have been made to use the results of systematic evaluation in course design.

Bates (1974), reporting experience at the Open University, points out clearly one of the major difficulties involved in establishing an adequate evaluation system, namely the fact that the creation of such a system means finding substantial resources which may not originally have been budgeted for. Understandably, the main aim of any organization creating a new educational system is "to get the system off the ground" and naturally, if there are financial pressures on the organization, priorities will go to course production, tuition, counselling and so on, possibly leaving inadequate funds with which to establish an evaluation system which is ultimately necessary if course teams are to improve their methods and materials.

The review of the experience of the Federal Republic of Germany with developing distance study systems (Heidt, 1975) brings out the point that some of the problems facing the designers of education systems stem from those who exercise political control over the system. The author makes the point that it is still not generally accepted that fixing objectives for a project is a political act and is an essential pre-requisite for the planning of a means to achieve these objectives. Heidt argues that the avoidance of taking the appropriate decision inevitably leads to problems and delays. For example, Heidt points out that the request by the authorities that a comparative study be made between the cost of distance learning and conventional study before making a decision on the introduction and form of a distance study system was impossible to meet without the objectives of the distance study system being specified. Experience at the Open University suggests that this problem extends downwards into the organization itself and can give rise to problems in the production of the courses of study themselves. For example, a whole range of difficult problems face the producers of mathematics courses if there is no general agreement within the institution about the goals for the education system as a whole and about how the mathematics component is supposed to aid the achievement of these goals.

An aspect of problem identification which does not, so far, appear to have been written about anywhere, is the use of "learning models" as a means of understanding student behaviour and identifying learning problems. This approach is, in my view, one which is appropriate to the teaching of mathematics and is probably one technique which can be applied to the systematic investigation of the pedagogical approaches to the teaching of mathematics. The work described by Poulsen (1975) is a step in this direction.

4.2 Improving a system

Once a problem has been identified, it must be removed if an improved system is to be created. This is a difficult task; the best that can be achieved at present is the introduction of systematic methods of searching for improvements within the constraints imposed on the system. Any systematic search for an improved system will depend on feed-back being collected in a reliable and systematic manner as described in the previous section. At the present time, all one can do is speculate on how successful any search for improvement will be, for no one appears to have written up the results achieved as a result of systematic improvements. At the Open University, we are just starting the process of producing "second editions" of our oldest courses and, consequently, are just coming to grips with the many problems involved in designing an improved system. Our experience suggests that the following factors need to be taken into account.

(a) *Specifying the objectives of the improved system*

The objectives of the original systems may not have been specified adequately. In addition, changing circumstances may make it appropriate to revise objectives before attempting to produce an improved system.

(b) *Influence of organizational and operational constraints*

It is important to decide whether the new system is to be subject to the same organizational and operational constraints as the old one and if not, to decide what the new constraints are.

(c) *Cost consideration*

It is important to consider the cost of system improvements and if resources are scarce to balance the benefit that may be obtained from an improved system against the benefits arising from alternative uses of the resource put into the system improvement.

(d) *Production of improved teaching materials*

It is important to ask if sufficient has been learnt about the production of teaching materials to be able to go ahead with any confidence on the production of improved teaching materials.

(e) *Problems of integration*

The multi-media systems like those being produced by the Open University and others are highly integrated, in that each component of the course depends intimately on the properties and characteristics of all other components. This makes it very difficult, for example, to improve one component, say the broadcast element, without making large numbers of changes to all other components. This problem presents real difficulties in the case of mathematics teaching where, for example, notation and terminology must be consistent throughout the course, so that it is not possible to improve the notation used in a television broadcast without making this notation consistent with that used in all other components of the course.

(f) *Gradual evolution versus total replacement*

One of the most difficult choices facing those who want to produce an improved system is to decide whether to attempt to evolve the present course into a more satisfactory one or whether to start again and provide an alternative course designed to meet the same goals and objectives. At present, our knowledge of how students learn mathematics is so poor, it may be safer to adopt the evolutionary approach, provided that the feed-back obtained on the original course is adequate to allow the course team to judge where to make improvements.

5 THE CHALLENGE AND A PROPOSAL

The authors of the Council of Europe Report (1975) assert: "it can be fairly said that distance study systems are achieving positive results. The best proof of this is that many of them began as short term ventures and have since been deemed worthy of being extended and institutionalised". They also point out that there is a substantial degree of uncertainty about the right form for such teaching systems, adding that this is not surprising in view of the wide variety of media and methods employed, many of which are original so that course producers are not able to draw on previous experience.

Clearly, the purpose of introducing discussion on a novel topic like adult and continuing education in mathematics at an international congress on mathematics education and including a chapter on that theme in this volume is to allow the producers of adult education in mathematics to exchange their experiences so that they can learn from each other. In these circumstances, this chapter should serve as a source of reference giving clear and accurate information on activities in the field covered by the report. Unfortunately, as pointed out in various places in the chapter, the current state of the literature in the field is unsatisfactory. While papers have been written on adult and continuing education and particularly on the design of distance learning systems (for example the Council of Europe Report (1975) contains a bibliography of 187 items, most of which were published in the last five years) little appears to have been written on the teaching of particular subjects in adult and continuing education. For example, of the 187 items just mentioned only 10 appear to have an explicit subject content and none addresses the teaching of mathematics. Thus the challenge to be faced by all those working in the field covered by this chapter is that of documenting their experiences and of finding ways and means of making these experiences known to others. Below is a proposal designed to meet this challenge.

In order to develop the material on this theme further, it is necessary to obtain information on the aspects mentioned below.

- Adult and continuing education programmes in *mathematics* which are being undertaken in various countries and any major factors influencing the development of these programmes.
- Approaches to the teaching of mathematics in adult education and in particular on difference of approach depending on whether mathematics is taught as a subject in its own right or as a topic integrated into broader programmes.
- Examples which illustrate approaches to teaching mathematics in adult education and which serve to show which approaches have been successful and which have not.

The more detailed and specific the information that is provided, the more useful it is likely to be in extending and improving our knowledge of the subject. In this context, it is particularly important to know whether a project has been evaluated, the circumstance in which the evaluation was made and the results obtained.

Sensibly, an international body concerned with Mathematics Education, such as ICMI, should seek to establish an international centre to collect and analyse information on adult education in mathematics. The function of such a centre would be to:

- gather, organize and disseminate information on existing projects and on educational resources created by these projects;
- encourage the exchange of information on educational technology and research techniques appropriate to the teaching of mathematics in adult education;
- promote co-operation in the production and exchange of educational materials for mathematics teaching in adult education.

Initially such a centre does not need to have any formal "legal status", although there are no doubt certain administrative and supervisory arrangements which would have to be taken care of, and limited finance to be provided. To start with, such a centre could be established to act simply as a means of providing co-ordinated information about the teaching of mathematics in adult education to be used for example at the next Congress on Mathematical Education.

While this proposal is an ambitious one and will certainly involve a great deal of work for whoever takes it on, without an initiative of this sort the discussion on adult and continuing education in mathematics at the next Congress will be based on the same sort of sketchy and inadequate information as the discussion at the 1976 Congress.

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Teaching at a Distance.

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PANEL

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universal and automatic solutions to the large spectrum of international problems in question. On the contrary, anyone confronted with such problems has to seek specific solutions tailored to his own situation. The purpose of such a report can only be to provide cues for orientation at a general level by furnishing a conceptual framework for the contextual analysis of problems.

An analysis of one's own situation considered within the suggested framework and along the lines of the issues raised in this chapter might be of help for structuring, ordering and clarifying one's own views. In order to get a clearer picture it is often useful to take a step backwards.

This report will assist those working in the field to do so. This also means that this report cannot provide immediate recipes. It has been our major aim to cater for communication with respect to the problems of mathematics teachers and to further the necessary co-ordination of all efforts made in this domain.

Communication also requires the observer to adopt a detached attitude, and to be willing to generalize to some extent from his own concrete situation. We think that these goals are in harmony with the recommendation adopted unanimously at the final session of the section for our theme at the Third International Congress on Mathematical Education and addressed to the Executive Committee of ICMI, according to which "the education and the professional life of mathematics teachers" should be accepted as a permanent theme for future congresses.

2 THE EDUCATION AND THE PROFESSIONAL LIFE OF MATHEMATICS TEACHERS

2.1 Teacher education and school reform

Though very little attention has been paid to the education of teachers in the framework of educational reforms, the former has been decisively influenced by the latter in two respects: first, the growth of educational opportunities necessitated a great expansion of teacher-education institutions; second, the introduction of new curricula, new classroom technology, and new forms of organization in everyday school life forced changes in the content and organization of teacher training.

The education of teachers today is, like other parts of the educational system, affected by limitations in financial resources. These limitations are concomitant with increased feelings of resignation concerning educational reform. "There is the conviction that complex social problems are not as tractable as government planners thought in the 1960s. For whatever reasons, the shiny plans do not result in many heartening practices; the dreams turn sour." (OECD, 1974g, p. 4). This does not apply to all countries equally, since the educational systems and, consequently, teacher training in many countries are less subject to economic and ideological fluctuations as a result of the longer range and greater stability of nationwide planning.

But criticisms have also been voiced, in connection with a widespread disenchantment with educational policies, which attribute the deficiencies of past reform efforts to the insignificant role teachers were allowed to play in them. The statement applies also to teachers that ". . . the readiness of the users to implement new structures, products and practices is more of a problem than the development of a particular product itself" (Dalin, 1973, p. 260). The importance of teachers in educational reforms, their ability and willingness to innovate, the amount of information available to them, their participation in decision-making processes concerning the goals of innovation, as well as the organizational setting of their activity, are stressed in all studies of innovation. However, this has not been taken into account in educational reforms, as can be shown for instance by the secondary role of teacher education: "It could well be argued that teacher education should take the lead in the development of an educational system and that it should be given priority in the social and educational measures to be taken by States. However, the history of education shows that in practice teacher education has hitherto been dealt with only in an *ex post facto* manner." (Kotasek, 1972). This assertion is especially true for mathematics education. The Unesco report published in 1972, *New Trends in Mathematics Teaching*, vol. III, for instance, contains only scattered references to the role of teachers and of teacher education and no chapter specifically dedicated to the topic. A change in orientation has occurred in recent years concerning the significance and structure of the teacher's role. The importance and relative autonomy of teacher education, therefore, is increasingly recognized, and the possibility of introducing innovations into the school system through the channel of teacher education has been considered: "The reform and continuous development of teacher education is directly dependent on reform and developments in the school system and education as a whole. Teacher education can never be autonomous or independent. On the other hand, changes in teacher education are not to be viewed merely as necessary consequence of changes in the school system and the field of education. Teacher education is also a developing motive force which can lead to changes and improvements in the educational system." (OECD, 1974a).

Not the least influences favouring such an approach are the aforementioned tendencies to reduce financial expenditures for education and its reforms. Since 70 to 80 per cent of all educational expenditures are spent on teaching personnel, whereas only 4 per cent are spent on books and other teaching and learning aids, efforts to increase the effectiveness of the teaching force are obviously of great economic significance. As the scarcity of resources is unlikely to promote the establishment of new institutions in the foreseeable future, we will depend heavily on the existing network of pre-service and in-service institutions in our attempts to optimize the performance of teachers. Even the trend which, according to the NACOME report (1), is likely to have some impact on U.S. teacher education in the near future, i.e. the introduction of Competency-Based Teacher Education (CBTE), owes its dynamics and influence to cost considerations and the resulting pressure on the educational system.

Not only does pre-service education currently play no active part in educational reform, but it is often unable either to respond to changes in actual school practice or to prepare teachers for the present requirements of the profession. In many countries, the heterogeneous character of the school system still constitutes the major obstacle to the design of a practice-oriented mathematics teacher education. The last few years have witnessed changes in most countries that have decisively influenced the professional life of mathematics teachers, inside and outside the classroom. As far as the curriculum is concerned, these changes tend not to be centred on the subject-matter quality of the programmes. "Responding to the concerns of classroom teachers, as well as educators and laymen interested in the basic goals of general education, attention has now shifted to programs for less able students, to minimal mathematical competence for effective citizenship, to the interaction of mathematics and its field of application, and to the impact of new computing technology on traditional priorities and methods in mathematics. Furthermore, the dominant role of mathematical structure in organizing curricula has been challenged by many who advocate pedagogical or psychological priority in determining scope and sequence." (NACOME report, 1975, p. 23).

Changes are especially noticeable in the composition of the student population, the climate of interaction in schools, and the organization of classrooms.

In most countries, changes in the attitudes and behaviour of pupils have made teaching more difficult. Reports from the U.S.S.R. say that children today are more eager to learn, more critical of authority, and make greater demands on the teacher's subject-matter knowledge. In many countries, discipline problems have become more acute, especially in schools whose pupils come from underprivileged sections of society. In some countries, the introduction of comprehensive schools, accompanied by a greater diversity of achievement, has confronted secondary teachers with new problems, to which pre-service education is gradually trying to adapt.

Teachers are increasingly forced by pupils and parents to justify their teaching also with respect to the selection of content and the relevance of mathematics for the pupils' future life. The patterns of rapid change in school norms are at best noted in the pedagogical components of teacher education, whereas content-oriented instruction usually pays insufficient attention to this problem. The influence of hand calculators, now cheap enough for some pupils to buy, the availability of computers, and the switch to the metric system in the United States have raised classroom problems for which teachers are insufficiently prepared by their pre-service education. As a consequence of such developments, the social problems of mathematics teachers, who teach a subject which is rather unpopular but considered very important, have become more acute in and outside the schools.

Not only do these problems show the special importance of teachers in educational reforms, but they also shed light on the increasing importance of social and economic factors in education. In the near future, more comprehensive educational analyses and evaluations, which will include these factors, will undoubtedly prevail over spectacular reform efforts directed towards single aspects of the educational system. More attention than before will also be paid to these factors in work concerned with school mathematics and the pedagogy of mathematics.

2.2 Environmental conditions of teaching

The changed emphasis in recent years concerning problems of teacher education and of teaching activities has occasionally caused an exaggerated consideration of the teacher as the critical factor. A conference report published by the OECD in 1975 states that "... the critical variable that distinguishes successful from unsuccessful schooling is not so much whether or not schools are comprehensive, unstreamed or have integrated curricula or are indeed characterized by any other administrative variable, but rather the capacity of teachers within them" (OECD, 1976b, p. 107 et seq.).

In contrast, there is a series of scientific analyses and reports of practical experience which stress more clearly the dependence of teachers' behaviour on the opportunities and limitations inherent in the context of their work (2). Elements of these environmental conditions or frame factors are, among others, the decision-making and organizational structures of the school system, the style of the school, the standards of colleagues, working conditions, working-time regulations, the curriculum, and teaching aids. These elements need to be clarified so that the teacher can consciously consider them and improve the quality of his decision making in his professional practice. Just as teacher education prepares teachers in the subject matter and methodology, so it ought to prepare them to cope with these demands.

We shall discuss some of these environmental conditions from two points of view: first, the freedom enjoyed by the individual teacher to make decisions in his professional life; and second, an adequate organization of the teacher's professional activities. The ways in which decisions are reached concerning the curriculum and the forms for supervising classroom practice vary considerably from one country to another (3). The consequences of these differences should nonetheless not be overestimated, because even in centralized systems the teacher enjoys relatively great freedom in the interpretation and implementation of curricula. Although his classroom decisions are sometimes seriously hampered by centralized examination regulations and standardized tests. A strong and uncontrolled influence is furthermore exerted by teaching aids. There is a lack of criteria for evaluating and choosing materials, and, accordingly, teachers fail to use them effectively. Support is also lacking for purposive problem-oriented experiments with new material, and the teacher himself may lack the necessary ability to do the experiments. Commercial textbook production in western countries is characterized by a continuous production of new material which shows little continuity, although it rarely varies in methodological and pedagogical approach. Apart from the teacher's manual, there is insufficient practical material for teachers that could help fill the gap.

The work load of teachers is heavy and has been increased in most countries. Quantitative data are hard to obtain, as the job-related activities of teachers cannot be easily separated from the activities they undertake out of "personal" intellectual interest. Generally, teaching activities in the strictest sense have tended to diminish, whereas because of the low level of organization and the increasing complexity of the teaching profession, the expenditure of time on administrative tasks, for instance, has grown. Since criteria for planning and evaluation for schools are generally lacking, and since the structure of decision making and organization leaves little room for planning measures in schools aimed at solving specific problems, the allocation of, and cuts in, funds are handled in a relatively formal way. For the same reason, it is difficult to develop and enforce principles of a task-oriented work organization. In the U.S.S.R., where efficiency criteria are more often applied to the school staff and the school system as a whole than to the individual teacher, efforts have been made to improve the efficiency of teacher activities by developing a theory of the scientific organization of pedagogical work, whose findings will also become part of teacher education. Such a theory, of course, which must integrate knowledge from pedagogy, psychology, the psycho-physiology of work, space planning, and economics, has as yet only been sketched in broad outline. "In spite of the great number of methodologies in the different subject matters there is not even a scientific basis for the organization of teachers' work in any concrete subject." (Borisova and Turcenko, 1975, p. 46). Important aspects in this connection are a functional specialization and a correspondingly intensive co-operation among the school personnel. "Ideally the organization of teams of specialists within schools and their co-operation with others concerned with the socialization of the young in the community will come to be seen as a reinforcement of the teacher as expert educator and consultant, advising others on the basis of his specialist knowledge and co-ordinating their efforts." (OECD, 1974b, p. 25). The proposal for a "supplemented teaching force" (Renshaw, in Lomax (ed), 1973) is making slow headway in the daily life of mathematics teachers, but it is unavoidable in the long run, for economic reasons among others (cf. OECD, 1976, especially p. 57–58).

2.3 Professional orientation in mathematics teacher education

Recent discussions of the reform of mathematics teacher education have increasingly considered items related to professional practice. The difficulties met by teachers in their daily professional life loom much larger.

Evidence of this is given by numerous reports on the experience of teachers starting their career (4). Although pre-service education is, for the most part, not responsible for these problems, which it cannot solve either, many prospective teachers have been seized with feelings of uncertainty concerning their approaching professional practice, which makes corresponding demands on their education. "Practice orientation" has become a catch phrase covering a whole range of positions from the students' need for an immediate preparation for practice to the interest of the educational administration in a money-saving reduction of education components that are not of immediate professional relevance.

What professional relevance means for mathematics teacher education specifically is unclear. The contribution that the individual disciplines (mathematics, pedagogy, psychology, sociology, philosophy, etc.) should make to the professional knowledge of teachers is as uncertain as the appropriate relationship between the skills needed for dealing with current practical situations and the knowledge and orientation needed for a long professional life in a rapidly changing school. A few new universities in western Europe have made practice orientation – in conjunction with such principles as interdisciplinarity and "project studies" – the cornerstone of university education, thereby gaining experience that sheds light on the complexity of the problems.

The non-mathematical education of mathematics teachers is characterized by a great variety of designs. Whereas there is a tendency in secondary school teacher education to put more weight on non-mathematical components, efforts to bring training institutions for primary school teachers up to university standards have made it difficult to find an adequate mixture between scientific education and practice orientation. The "Field Based Teacher Education" movement, which tries to intensify "student teaching" and co-operation among schools and universities, is gaining ground in the United States and is important for mathematics teacher education. In other countries, however, the practical parts of teacher education have been reduced in favour of theoretical courses in educational sciences.

A description of quantitative trends in mathematics teacher education relative to professional practice does not, of course, portray the content problems raised by such a development: i.e., the determination of an adequate concept of practice for the teaching profession, the establishment of qualifications a teacher should have, and the construction of models for acquainting the prospective teacher with the problems of his/her professional activities. The relation to the content taught presents a special difficulty when dealing with these problems.

There is a widespread tendency, especially in secondary schools, to identify the professional life of teachers with immediate classroom activities, and to underestimate activities related to organizational questions and contacts with the environment. In some countries, however, as an expression and consequence of a higher awareness of the importance of education for social development, efforts can be noticed that are "aimed at removing the cultural myopia of teachers, which is often the result of their narrow schooling . . . As far as the content of courses in teacher education is concerned, the need is to accelerate the introduction of evidence from the social sciences and the social context outside of schools" (OECD, 1974a, p. 21). Such intentions have been pursued, for instance, in large-scale school-oriented experimental reform programmes; but, in general, there has only been a gradual increase in the share of social sciences in the pedagogical training in western countries. In communist countries, all teachers take comprehensive courses in social science during their education and once in service participate in cultural and social activities outside the classroom. The difference in scientific methods and interpretations between natural sciences and social sciences in western countries presents additional problems for mathematics teachers. The problem of the cultural isolation of mathematics teachers does not seem so large in communist countries.

The broader concept of practice has certainly added substantially to discussions of the components of mathematics teachers' activities; however, no substantiated analysis of school teaching and of its environmental conditions, no analysis of the daily activities of teachers and of the freedom of action they enjoy in their everyday work, have been produced so far.

For teachers, there are no internal standards of professional competency as in other professions, since the question of the aims of teaching cannot be answered by the teaching profession alone. Thus, the insufficiencies of the empirical data on, and of the conceptualization of, school practice is a major obstacle to the development of precise and unified ideas about what relevant qualifications are desirable for a mathematics teacher. As far as teacher education is concerned, the *perspective of teaching* and the development of a theory of teaching are more important than the perspective based on the hitherto more sophisticated theories of learning. Besides the problem of providing empirical data on teaching, there are other more important prerequisites for a conceptualization of a teacher education that would be more relevant to professional practice. There is a lack of coherently formulated ideas about the relationships between cognitive knowledge, emotional dispositions and teacher behaviour from which criteria for theoretical and practical qualifications and their inter-relationships could be derived. Properly worked-out ideas about the relationships between the different components of the teacher's theoretical knowledge that are relevant to practice, as well as theoretical concepts and empirical knowledge about the process of acquiring professional competency and its optimal design, are also missing. These shortcomings are reflected, among other things, in the prevailing unstructured multiplicity of offerings and in the lack of co-ordination of the courses offered. As a result of this, students are confronted, even within the different disciplines, with the great number of highly different and partly incompatible conceptions the various educators have of the educational context, and the elaboration of a meaningful whole is left to the individual student (Becker, 1975). Whilst the problem of relating theory and practice in mathematics teacher education is essentially a problem of the global structure of the education programmes, recently efforts to enhance the links with practice have been predominant in teacher education.

In the field of educational methods, for instance, new models have been constructed to acquaint the student with the professional activities of teachers. Attempts are being made to develop, gradually and systematically, conceptual and diagnostic skills as well as behavioural strategies for handling real situations. This is achieved by different ways of simulating real situations, working with reduced complexity. Micro-teaching, special forms of training in classroom observation, group dynamics, protocol materials, and other training methods are increasingly used and are being developed further (5). Since the large majority of these techniques are based on principles that are independent of subject matter, they play at present a minor role in the professional education of mathematics teachers. Moreover, it remains true that "the missing factor is valid content . . . Knowledge of *how* to train teachers has outstripped knowledge of what to teach them" (Smith, 1975).

In view of the above mentioned deficits of the theoretical prerequisites for the development of practice-oriented teacher education programmes interrelating systematically the different components of teacher education, various efforts have been made to provide a basis for the internal co-ordination of education programmes. So, one of the most prominent, and also one of the most controversial, reform movements in the field of practice-oriented teacher education considers the gearing of education programmes to a unified set of goals as an example of such an organizing basis. Competency-Based Teacher Education (CBTE) is an attempt to develop catalogues of teacher competencies through institutionally controlled planning and goal setting processes. These specified competencies provide a basis for teacher education. CBTE is based on " . . . the desire to make schooling more efficient and to hold teacher educators and teachers accountable for the effects of their efforts" (Joyce, 1975). For all the efforts involved, research on the characteristic features of effective teaching have so far produced few results that are of practical value for CBTE and that could justify its claim. However, the CBTE movement has achieved some success in the organizational and content co-ordination of teacher education; nevertheless, a final assessment cannot be made before a systematic analysis of the innovation processes involved is available.

That the scientific determination of a teacher's necessary professional qualifications will probably remain an unsolved problem for some time has given new impetus to another current of teacher education, which considers the individuality of the teacher as the starting point for designing programmes. "The teacher educator who emphasizes the uniqueness of the child and the importance of his emotionality must do the same for the teacher ... The effort of the teacher educator has to focus on the development of a rich, many-sided, actualizing self rather than on training the prospective teacher in specific skills which will be shared with other teachers." (Joyce, 1975). This "humanistic" approach to teacher education can be found in one form or another as a counterweight to behaviouristic conceptions in most western countries, especially in Scandinavia and the United Kingdom. Practical ideas about possible forms of mediation between the preparation of teachers to help them meet the objective demands of teaching and the necessary development of the "subjective factor" are still missing. The different forms of a more practice-oriented education that we have sketched are aimed at improving the individual qualifications a teacher brings to his activities. But it should not be overlooked that "... excellent achievements in pedagogy are not made possible by personal abilities and creative independence only: but also because of a high level of organization in everyday work" (Borisova and Turcenko, 1975, p. 45).

2.4 Problems of professionalization

"The improvement of mathematics teaching is a matter of improving the professional quality of the mathematics teacher" (Fletcher, 1975). A profession is usually characterized by the following features: first, the members of a profession have highly specialized knowledge and have received a specialized education; second, the members of a profession are held together by strong corporate bonds, which facilitate the development of a "collective wisdom", namely, a common stock of knowledge gained from experience; third, a profession enjoys a certain degree of sovereignty in its field. In each of these three characteristics, the teaching profession has specific difficulties that distinguish it from other professions.

Since the activities of teachers are close to the educational tasks performed by other groups (parents, mass media, etc.) and since society has a special interest in socializing the young generation, the autonomy of the teaching profession must be determined by the relationship between control by society and control by the profession itself. The influence teachers can exert on their profession and on the conditions of their education is highly dependent on whether the teaching profession develops its own self-understanding. The two other characteristics of a profession play a decisive role here.

The corporate character of the teaching profession is most clearly expressed in the organization of teachers in unions and professional organizations as well as in the nature of the daily co-operation between the teachers of the same school or of the same school district. The degree of organization of mathematics teachers in unions is a great deal higher than in professional organizations and varies between 50 and 100 per cent in most countries. In some western countries, teachers unions have become more active in recent years and have tended to dedicate themselves increasingly to issues that go beyond the traditional union concerns (e.g., education and admittance conditions; educational policy; teaching problems); conversely, many professional associations have had to concern themselves increasingly with trade union problems. But building a common body of professional experience is more a problem of immediate on-the-job-co-operation between teachers, and this is a point where the diversity of school types raises a big problem in many countries.

The central point in the development of the teaching profession is the nature of the special knowledge it commands. On the one hand, some scientific disciplines are concerned with various aspects of teaching and of teachers' qualifications; but on the other, their specific contributions to research, development, and teacher education in the field of mathematics teaching are not precisely enough determined. Compared with other professions, the special structural problem of the teaching profession is that it does not have a "basic science" such as law for the lawyer, medicine for the physician. It should not be overlooked that scientific theory is related in two utterly different ways to the practical work of mathematics teachers: first, scientific knowledge and methods are the subject matter of teaching; second, the conditions and forms of its transmission must be scientifically founded (cf. Mies et al., 1975, p. 71). Thus, teaching is under a far more complex pressure than other professions to justify itself against competing conceptions of scientific theories, and has to cope with far greater demands with respect to the integration of diverse dimensions in the unity of action.

The increasing demand made upon the scientific foundation of the content and form of teaching has led in all countries to attempts to raise the scientific qualifications of the teaching profession. In the U.S.S.R., for instance, the proportion of university trained teachers rose from 14 to 60 per cent between 1960 and 1973. The "scientification" of education has been achieved, among other ways, by integrating institutions exclusively devoted to the education of primary school teachers into universities, polytechnic colleges, and similar institutions. True, this contributes to reducing the social isolation of the teacher; but, on the other hand, it renders the development of a specific professional profile more difficult although the content and formal aspects of the education given to the different types of teachers tend to become more similar. These tendencies have not emerged in all countries and are partially obstructed by efforts of administrative agencies to avoid lengthening the teacher education course, which is short compared with other professional training. We may say, however, that the teaching profession is gradually becoming a unified "all-graduate profession".

Although the "scientization" of teacher education may diminish status problems, it may increase institutional and communication barriers between science and school practice. "In fact, the teacher education function has been deemphasized as these colleges and universities have attempted to emulate the broad-based institutions of higher learning. Arts and sciences have been stressed, rather than the profession." (OECD, 1974g, p. 7). The controversy concerning this general trend toward a more scientific teacher education stems from increased demands made on the integration of the various forms of theoretical and practical experiences. "In contrast with certain other professions, that of engineering to take but one example, it is striking to observe that the preparation for the job of being a teacher is not organized on a basis of the interrelations between research, training and educational practice, nor is it oriented in that direction. A major reason for this state of affairs is that it is very difficult to integrate these three components; in fact the serious education of teachers is a problem whose nature is highly complex: it is essential to create permanent links, on the one hand between scientific research and teachers, and on the other between different disciplines." (Glaymann, 1976) (Trans.).

One of the most important factors influencing the development of the teaching profession lies in the professional opportunities for teacher educators, the "teachers of teachers". This applies to the teaching staff in the related disciplines, such as psychology, sociology, etc., which are not primarily concerned with teacher education, as well as to the educators in general pedagogy and to specialists in mathematics education. The necessary integration of qualifications in various scientific and practical areas is still contingent on the individual efforts of teacher educators. In fact, there is often a kind of Baron-von-Münchausen phenomenon: professional education must be conducted without a sufficiently differentiated infrastructure and without a common frame of reference for organizational and content issues, so that educators must acquire the necessary qualifications themselves through self-study; there are almost no in-service training institutions for educators. The excessive teaching load in most teacher education institutions scarcely allows for personal research on specific professional questions; what is available is scattered and does not readily merge into a common stock of theoretical insights. "It does not add up. We need more concerted, co-operative and collaborative efforts." (Osborne, 1976).

The relationship between teachers and research is fundamental for the development of a common professional self-understanding and for the internal coherence of the profession. Evaluations in a great number of countries show that educational research and research in the field of pedagogy of mathematics have virtually no influence on school practice and that teachers' attitudes toward research results are sceptical (Cane and Schröder, 1970). The gap between fundamental pedagogical reflections and practical teaching can be bridged by individual persons or institutions in exceptional cases only. The development of the profession is in a large measure dependent on what instruments of mediation can be devised at the levels of organization and content to secure gradual transitions and feed-back processes between theory and practice. Few of the existing institutions can perform such a function, and in most cases, they are insufficiently integrated into the system of research, development, counselling, and education. Such mediating agencies should not be designed as superstructures taking the whole problem of the development of the profession under their own guidance; they could function as a kind of catalyst for the necessary changes in the domains to be mediated. At all levels, the main shortcomings are in content. Even where there exists a developed network of informal contacts between teachers and educators, as in the United Kingdom, "it is more necessary to do fundamental work than to convey the obtained knowledge to colleagues" (Kempa, 1976).

2.5 Specific problems of in-service teacher education

"There seems to be international unanimity on the crucial part that should be played by continuous in-service education for teachers, for teachers of teachers and for others involved in administration in the education system." (OECD, 1974a, p. 519). But this massive demand (6) for greater efforts in in-service education is in sharp contrast with the fact that in most countries opportunities to complete this task are badly wanting. Very few countries have in-service education schemes with corresponding programmes that are compulsory for all teachers. In the following, we shall have to bear in mind the extreme diversity of forms, goals and resources of in-service education (as compared with pre-service education).

Some of the main arguments stressing the urgency of in-service training for teachers – such as the short duration and the shortcomings of pre-service education, the dynamics of professional practice in the course of a long professional life, the quick changes in, and expansion of, the required qualifications – have been in existence for a long time. The present trend is therefore not easily explained by them. Two reasons seem to be decisive at present.

On the one hand, there are perceptible tensions between "theory and practice", between institutions concerned with teacher education and the schools themselves. Given the lack of influence of teachers on university education courses and a certain suspicion by many practitioners towards "theoretization", one expects relevant improvements from an in-service education that is more under the control of the profession itself (Fantini, 1971, p. 192). In many countries the school districts and teachers' organizations rather than the universities and teacher education institutions are responsible for in-service education. "In-service education is moving quite rapidly to the control and design of local public schools." (DeVault, 1976).

On the other hand, we have now reached an important juncture: the relatively large curriculum reforms of recent years have shown the necessity for in-service education (7).

In most countries, this coincides with a rapid decrease in the average age of teachers, a consequence of the large-scale extension of the education system (8). As a great number of the teachers of the year 2000 are practicing now (assuming the demand for teachers remains constant) and in view of the great mobility which is expected from teachers, as well as from other professions, there is going to be a greater need for in-service education in the future.

An outstanding characteristic of current in-service education is the disconnected variety of course offerings and the lack of continuity between courses. "In-service training for teachers must cease to have the haphazardness of a department store selection, and ought to be conceived of as an integral part of teacher education, whose conceptions will have to be developed in every subject." (Hahn, 1974, p. 79).

The institutional system of in-service education is also uncoordinated and fragmented, with efforts that are often random and not in harmony with each other. "In sum, in-service education has been the weakest and most haphazard component of teacher education. Even the most charitable would have to admit that it has not been nearly as effective as it might have been, considering the expenditure of time, efforts and resources." (Edelfelt and Lawrence, 1975, p. 16; see also Meade, 1971, p. 211). In some countries (e.g. the United States, the Federal Republic of Germany), the lack of unity in in-service education is such that several organizational structures exist side by side, with institutionalized and centralized forms in some regions and only informal efforts by travelling agents of dissemination in others. The fragmentation of administrative and economic responsibilities is not good. In-service education activities are generally conventional (lectures, seminars, self-study, etc.); thus, their efficiency is highly dependent on the organizational infra-structure in the schools (e.g., on the degree of co-operative preparation of courses and follow-up activities). International experiences seem to show that the higher demands made on in-service education require a greater willingness from the teacher and better opportunities for co-operation and communication among teachers.

Beside the usual courses and seminars, opportunities for in-service education may occasionally be offered in different forms:

- paid full-time study, with the possibility of obtaining a final degree (at the Cambridge Institute of Education, for instance, or similar institutions in the United Kingdom);
- individual counselling of teachers at their request (e.g., in the U.S.S.R., or in the teachers' centre movement in the United States or the United Kingdom);
- permanent teams of teachers, who sometimes set themselves ambitious tasks: the development of curricula and teaching aids, the organization of scientific conferences, etc. (e.g. Galion in Lyon; cf. Carrier and Gauthier, 1974).
- systematic correspondence courses (e.g., in the U.S.S.R.).

France is an exception, as far as the in-service education of mathematics teachers is concerned. In this country, 25 institutes (IREMs) have been founded since the end of 1968; these institutes, spread over the whole country, concern themselves with research on and development of mathematics teaching, and especially with in-service teacher education. "The experience of the IREMs, linked to universities, shows clearly that it is with the aid of the University that one can solve the problem of the education of teachers." (Glaymann, 1976) (Trans.). The IREMs combine a unified organizational structure with a high degree of local autonomy. "Although in France all the IREMs have *the same structure,* they have however a total autonomy in the organization of in-service education and research. It is just the great diversity which best allows them to cover the whole range of the realities of mathematics teaching." (Glaymann, 1976) (Trans.). In France, the IREM system is considered to be very efficient and a great advance. This example shows, however, that substantial efforts must be made to achieve measurable success. One of the greatest problems seems to be the training of qualified educators. Other countries report similar experiences: "Great efforts were made to prepare and maintain the dissemination agents. For 1½ years we met nearly 5 hours a week to acquire additional knowledge in the mathematical subjects and to share experiences", says a report from the Federal Republic of Germany (Kühl, 1975).

All in-service teacher education programmes are torn between practical demands and the wider horizon of a theoretical perspective: "If priority in the selection of subjects is to be given to their reference to reality, we should simultaneously make sure that the shortcomings of the existing system are not consolidated by regression to a merely pragmatic practicality. Thus, the regulating system of supply and demand can become a criterion for the selection of subjects only if the whole design is controlled by theory. Besides, this will help us avoid plans that would miss the needs of the users." (Kramp et al., quoted by Hahn, 1974, p. 80).

A large-scale opinion poll (more than 2000 usable answers) conducted in the United States by the National Council of Teachers of Mathematics (NCTM) shows that primary and secondary school teachers unanimously give the highest priority to "problems of attitudes toward mathematics" (including "motivation problems") and to problems of "application of mathematics" in in-service education and, what seems to be more essential, that in these same fields the widest gaps are to be found between supply and demand in in-service education (cf. also the NACOME report, 1975, p. 91 et seq.).

British opinion polls (cf. Cane, 1973) have produced somewhat similar results (e.g. all secondary school teachers have given great priority to pedagogical and organizational issues over information on the instructional subject). But a more significant point in this connection is that, as in the United States poll, both teachers and teacher educators find it difficult to identify problems with sufficient precision and to design in-service education accordingly.

If we had to give a short résumé of this chapter, the following point would have to be particularly stressed. Pre-service education, school reform, the limited capabilities of teachers, in-service education and the development of the profession all denote various dimensions of a complex system which, among other things, has to be considered as the background of all deliberations about the improvement of teacher education. This does not imply that immediate changes are not possible or that the corresponding plans for improvement would be useless; but it means that the interrelationships within the whole system must be carefully taken into consideration in such attempts. Neither can conceptualizations of parts of the system be meaningfully developed that do not take the achieved stage of development of the whole system into consideration, nor can significant changes be brought about without the starting point of the changes being precisely identified in its connections to the whole system. So there can be no productive thought or action about the education and the professional life of mathematics teachers, without the realization that these form an indivisible whole in which both of them simultaneously play an active and interdependent role. This implies that neither the education nor the professional life of teachers can go very far in their respective development without making reference to the other component.

3 MATHEMATICS AND TEACHER EDUCATION

3.1 Mathematics as a scientific discipline and as a school subject

The share of mathematical instruction in the education programmes for *primary* school mathematics teachers has tended to increase in the period under review. Some content areas previously reserved for the education of secondary school teachers at universities, such as linear algebra and transformation geometry, have partially found admittance in courses for primary school teachers. Other components, such as the widespread foundation courses in so-called set theory, formal logic, and combinatorics, have also been introduced in harmony with developments in mathematics education for primary schools. This evolution is in line with the formation of the principle of specialization for primary school teachers and with the interest "in linking the training of primary school teachers to higher education . . . . . The specific problem of 'normal' education (education given in teacher training colleges and similar institutions) is its position in relation to university education" (OECD, 1974a; which does not apply to OECD countries only). Subject matter specialization and "academization" have always been striven for in primary school teacher education on the grounds of social status (cf. for England, Hewett, 1971, p. 17 et seq.). In many countries, however, a great number of primary school teachers have received no mathematical education whatsoever. In other countries, mathematical education is compulsory for primary school teachers.

As far as *secondary* education is concerned, the following statement from the NACOME report (p. 85) seems to characterize well the general situation: "The senior high school teacher's content preparation is little changed from the 1960s except that more recent graduates are more likely to have worked with computers, are more likely to have taken courses in probability and statistics and perhaps combinatorics, and may have been exposed to some serious work in applications or modelling. Given that more of them will teach geometry than any content other than algebra or general math, it is likely that the weakest link in their content preparation is geometry." An exception to this statement is the U.S.R.R.: geometry has represented approximately 25 per cent of the mathematical instruction given to upper secondary school teachers, and it will still make up 15-20 per cent after the implementation of proposed changes in teacher education. A controversial content area apparently given a higher priority by teachers themselves than by the university is logic.

The share of mathematical education in pre-service education at the university has remained unchanged or is decreasing. In general, it never amounts to more than 50 per cent and in most cases is less than one-third of the total time for the education. For instance, in the U.S.S.R. even a prospective mathematics teacher with only one subject does not give more than 45 per cent of his time to the scientific study of mathematics.

This partially benefits the extension of second subject studies (in many countries, the study of a second subject is compulsory: on the average, 50 to 75 per cent of the teachers have a second subject) and is due, among other things, to the increasing importance of the professional components of teacher education. "Secondary school teachers are traditionally university graduates and receive no real pedagogical training. This situation has been changing since 1965 to meet the new problems of mass secondary education." (OECD, 1974b, p. 13).

Comparing the major weak points of the education of both groups of mathematics teachers, the same OECD survey makes the following statement: "One group needs a deeper academic education, the other a deeper professional training". But even at the secondary level, mathematics teachers have not always received a special university education in mathematics.

A teacher education scheme taking seriously its claim to be practice oriented cannot leave it to the pedagogical components alone to relate the content of the programme to the future professional life of teachers. How does this apply to the mathematics teacher? How, in particular, is his mathematical knowledge related to the content of the mathematics school curriculum and to mathematics as a science? What influences, relationships, and possible differences in the choice and interpretation of mathematical content result from a "teaching perspective" in schools providing a general education? How are the development and the transmission of professional knowledge organized and secured in mathematics teacher education?

University faculties of mathematics offer virtually no special lectures or seminars for teachers. The mathematics teacher usually studies just mathematics without reference to his future profession. Diametrically opposed to this is the tendency to conceive of the education of mathematics teachers as a mere transmission of details and recipes regarding school mathematics, or at best as mere curriculum instruction. This tendency overlooks the specific differences between the learning situations of the student and the future teacher. This contradiction indicates both the little concern shown in discussions of mathematics education for the specific problems of teaching, and the insignificant part played recently by teacher education in the framework of education reform. Changes in teacher education have mainly consisted of makeshift adaptations to the requirements made by the curriculum reform. It is no wonder then that teachers still stick to textbooks. Textbooks are, except in their sets of exercises, implicitly and covertly designed for teachers, not for pupils.

Put in an extreme form, the view that the teacher should actually be a research mathematician (a statement made especially for secondary schools) is sharply opposed to the view that he should have a knowledge of only the mathematical content to be taught. The following statements by Fletcher may thus claim a relatively general validity: "Is there any special knowledge of mathematics which the mathematics teacher has which distinguishes him from other mathematicians? It seems to me that the answer implied by many courses which prepare teachers of mathematics is 'No', whereas the answer should be 'Yes'. The mathematics teacher requires a general knowledge of mathematics in order to be able to communicate with other mathematicians, and also to establish his credentials; but he also requires special knowledge of certain areas of mathematics, in the way that an engineer or an astronomer requires special knowledge. In fact, all professional mathematicians have both a general knowledge and their own particular mathematical expertise. It is part of our problem that the teacher's special mathematical knowledge is inadequately defined and insufficiently esteemed." (Fletcher, 1975).

Reports from secondary school practice show that young teachers often turn to one of the following extremes: either they completely forget their mathematical studies and try to connect their interpretation and treatment of the subject matter directly to their own school experience, or they attempt to transfer the style of their own mathematical studies immediately into school practice – to such an extent at times that one can easily tell under which university professor they studied.

There are not only quantitative aspects to the problem of the special mathematical qualification of teachers: even though the overwhelming majority of experiences seems to confirm the plausible assumption that mathematics teachers with a better mathematical education prove more flexible and open-minded in their classroom behaviour when dealing with materials as well as when interacting with pupils, the relation is not one of simple linear proportionality (cf. Begle, 1972). As is the case for many other relations between teaching strategy and teaching results (cf. Brophy and Evertson, 1976), here there is also a kind of "inverted-U relationship" (9). There is, however, remarkably little consensus as to what mathematics a teacher should know, what his advantage in mathematical knowledge over his pupils should be, and what degree of specialization is tenable. The unsolved problem of the relation between university mathematics and the mathematics needed by a teacher in his professional life has generated controversial views among teachers, too. A comparative survey of probationary (*Referendar*) high school teachers' attitudes in the Federal Republic of Germany shows that in all other subjects "the degree of consensus on the education in the subject matter is *substantially greater* than among students of mathematics and natural sciences. In the latter group there is clear disagreement among trainees with a negative assessment of the studies in the subject matter and those who consider them highly relevant for a teacher's professional life" (cf. Frech, 1976, p. 77–78). Much the same is reported in a British opinion poll (cf. McLone, 1973, p. 71).

The controversy over the role of the content of teaching and the importance of the conception of mathematics for the organization of classes remains highly significant, as is the issue of the relation between scientific and school mathematics. These issues appear today as unsolved and problematic as ever (10).

Looking more closely at the problems that have been pointed out, one notices that the structure and content of school subjects are essentially determined by the scientific disciplines, whatever the development of school subjects, of teacher education, of teacher self-concept, etc., may be. On the other hand, the present development of technology and science is rather stormy, both with respect to the development of their inner structure and to their external relationships, their applications in the broadest sense. The science of mathematics, especially, represents today a highly specialized and dynamically developing system having its own contradictions and uncertainties which can hardly provide *immediate* clues for mathematics teaching in schools of general education. The attempt to maintain meaningful links between the scientific level and education through a vast extension of educational offerings or through a mere focus on scientific methods is no longer possible. The fact that scientific and school practice develop in different directions was, though from a different view-point, one of the decisive motives for the extensive efforts of the large curriculum reform projects (and perhaps for work on mathematics learning in general) during the '50s and '60s.

In his report on the development of SMSG, Wooton comments, for instance: "The rapid growth in size of the secondary school classroom and the consequent new problems of the high school teacher contributed to his gradual estrangement from the research scientist. The scientist, for his part, could no longer find the time or the means to continue the dialogue because of the explosive increase in productive research." (Wooton, 1965, p. 3). Science, for its part, has an interest in the development of all human capabilities and talents, and it is in its own development highly dependent on a progressive mobilization of those resources. Science, has therefore maintained a traditional, so to speak, natural relationship to teaching, which the famous physicist Peter Kapitza characterized in these words: "The history of science shows that the scientists whose work has been most fruitful have always been those who had students and who worked together with them. . . . Thus, Mendeleev discovered the periodic system of elements as he was looking for a method to describe the properties of the elements so that the students to whom he was lecturing on the fundamentals of chemistry could memorize them better. The young Lobachevsky, who taught high school geometry at an adult school sought an adequate method to give his students an expressive explanation of the parallel postulate. In so doing, he discovered non-euclidean geometry . . ."

This attitude towards the problem of teaching and learning is highly relevant today; but it is not self-evident and does not automatically gain clear and positive recognition. From the perspective of education, it is rather a problem. It is, for example, consistent with the natural relationship between the scientific level and school teaching to consider mathematics as an active process and knowledge as action, as activity – a view which is shared generally today (especially in the pedagogy of mathematics). But this implies, because human cognitive activity is many-sided that an active attitude towards the social and natural reality is always given with the mathematical content, and that the content can only be learned and acquired in this context. Communication, especially, is not just a question of clear language.

But the pupil, the teacher, and the scientist have quite different perceptions of social and natural realities. The professional life of a teacher in school, his social role, and the environmental conditions of his work are considerably different from those of a university scientist. If one fails to take into account the social and institutional aspects in determining the educational content, in working out principles for the teaching of mathematics, and in a broader perspective, in developing relationships between schools and universities, and if one simply tries to extrapolate the natural philosophy of teaching of science, the connection is blurred and nothing but very vague ideas of "openness", "flexibility", "creativity", "learning how to learn", and the like remain (11). It is no wonder that conceptions of the profession and of mathematics teacher education that result from this evaporation of the relation between science and teaching into such a vague generality are correspondingly problematic, and that the real conditions spontaneously – and often most unfavourably – prevail against these, *per se*, very desirable principles (12). The relationship to science of the principles and conceptions in the pedagogy of mathematics developed against this vague background is distorted and abridged, and the specificity of the content is not sufficiently considered. We will consider later the incorrect alternative of "child-orientation versus subject-orientation" (13).

All in all, there is, on the one hand, an overemphasis on the intuitive and implicit aspects of the manipulation of mathematical knowledge in the teaching process, a certain vagueness and shapelessness that has caused teachers to feel somewhat uncomfortable and that has made too many demands on them. At the same time, extremely generalized conceptions – abstract as well as formal-logical – have spread, not least as a response to this feeling of discomfort and these excessive demands, especially in connection with the so-called "new math". Further, there is often a parallelism between the dubious belief that the necessary abilities will develop automatically in appropriate problem situations (or that they, at any rate, result from ad hoc learning, consequently needing no transmission) and an occasionally pedantic treatment of the so-called fundamental structuring principles of mathematics in the classroom, a dogmatically narrow view of concepts very similar to the old skill drill. Should not the problem that in France (and probably in other countries too) mathematics has turned more and more into an instrument of social selection also be seen against this background? "The selective function has been switched from Latin to mathematics", states the French mathematician Samuel (1974), and he gives corresponding reasons.

One should think also of the discussion raised in the Federal Republic of Germany in connection with the *numerus clausus* (14) and the unification of the *Abitur* (15) requirements, concerning hierarchies of mathematical content and how to handle them in assessing achievement. Here, too, short-comings have been disastrous, which can be attributed to the lack of an adequate conception of the possible concrete links between school and university mathematics.

Lastly, some even recommend that secondary school teachers at least should be led to the frontiers of research to help them acquire the spirit and intellectual habits of creative mathematicians, while the real situation calls for the following summary: "By confining teacher education within the organizational structure of the university, the knowledge explosion affects professional content in a converse relationship; that is, as more and more knowledge is developed a decreasing proportion of such new theories, ideas, findings, or means of expression will become part of the content of teacher education. In sum, current content is more the result of the power politics played by established departments and faculties within the university than new developments in knowledge." (Haberman, 1975, p. 319; also Griffiths and Howson, 1974, ch. 15, especially p. 219-221). Surely, this outline of the consequences cannot be exhaustive. But the problem seems obvious: the traditional ways of relating the scientific level and school teaching cannot provide a solid framework any longer and have sometimes dissolved into conceptual vagueness. The failure to explore realistic possibilities of relationships with science and the failure to organize what is practically possible and socially desirable has often led to a latent hostility towards science in school pedagogy.

The fundamental philosophy of teaching and learning which probably found its strongest expression in Bruner's *The Process of Education*, and which was committed to the scientific development of the subjects seems to have vanished – at least to a large extent – in most countries. What has replaced it is still unclear and lacking in unity. Nevertheless there is much support for the following judgment: "One can scarcely overestimate the importance of the *general development* of the pupils for their future activity after they have left school. The progress of science and technology in our country is too rapid for school education to keep pace with it however well it may be presented." (Zankov).

There is undoubtedly a link between the degree of differentiation in one's conception of knowledge and the attention one pays to the specific problems of the teacher. There is, then a connection between the degree of differentiation of the conceptions of content and the teaching perspective.

The increasing complexity of the links between science and schools of general education has caused traditional views and practices concerning these links to become obsolete and has raised new problems that are complicated compared with problems of learning. And even if problems of teaching are still underestimated, more attention has been paid lately in discussions of mathematics education to the paramount importance of the teacher and of his role for the development of education. Changes in the content of mathematics teaching in schools and changes in the conception of this content have always been closely related to new definitions of the teacher's role. Simplified conceptions of the content generally result in an abridged view of the numerous aspects of teaching and of the teacher's role. Conversely, any attempt to write out a new definition of the content of teaching and of its adequate development calls for new reflections on the role and function of the teacher.

"The most important pre-requisite for undertaking effective innovation in teaching is to find a serious compromise between current science and practice in schools. This pre-requisite demands a school organization which allows the teachers to share the tasks of mutual communication and above all to be able to co-operate, thus putting an end to the sterile isolation of the teacher in the exercise of his functions as an educator. Let us remark that co-operation between teachers demands effort and above all a conscious collective determination which goes well beyond the simple performance of professional duties." (Glaymann, 1976) (Trans.). Greater co-operation among teachers is considered in many quarters an essential precondition for the development of modern school teaching, but this requirement is rarely related to the content of the teaching process or to its adequate development (which also includes the relation to science).

The merely organizational or pedagogical approach is as insufficient, or even as detrimental, as a conception of the content that leaves out the institutional and (socio-) psychological aspects. One may then even fail to see the relationships that are central for the desired changes.

3.2 The nature of knowledge and the teaching process

René Thom, a French mathematician and invited speaker at the Second International Congress on Mathematical Education held in Exeter in 1972, said: "In fact, whether one wishes it or not, all mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics. The modernist tendency is grounded essentially on the formalist conception of mathematics – that which was classically expressed in the famous aphorism of Bertrand Russell, 'Mathematics may be defined as the subject in which we never know what we are talking about nor whether what we are saying is true'." (Thom, 1973, p. 204). In making this statement, Thom was considering the point that communication in mathematics teaching is not only a question of using a clear language, but also of forming a common meaning. We can assume that little philosophy of mathematics "in the sense of an active field of work producing significant new ideas" (Hersh, 1976) has been developed in recent years and that still less philosophy of mathematics has entered into mathematics teacher education or into the pedagogy of mathematics. This situation in mathematics is diametrically opposed to that of physics and other natural sciences (cf. Mies et al., 1975). What little has entered has consisted mostly of formal logic. That is why Hersh is right in supporting Thom in his view of the so-called "new mathematics", simultaneously raising the problem to a more general plane: "To attack formalism in pedagogy, one must challenge its philosophical base – the formalist picture of the nature of mathematics".

These issues have recently become topical from a different perspective as well, namely in connection with specific problems of teaching. The aforementioned problem of meaning emerges here twice. On the one hand, there is the view-point that since scientific concepts always include general aspects, the theoretical objects of teaching are neither visible nor palpable. Their transmission therefore cannot be carried out like, say, throwing a ball. The teaching of theoretical concepts must take this into consideration and requires, consequently, a conception of the role of the activity, of the intuition, of the language, and of the logic of concept-development in mathematics instruction. Another aspect is related to the problem of general education and of its social importance, or its function in the development of personality. As a consequence of the manifest and paramount importance of esoteric special knowledge and of increasingly differentiated qualifications, many countries have paid little attention for some time to the concept of general education, and, above all, to the urgent need to update it. Only recently has a change of attitude occurred, especially among mathematicians themselves, but satisfactory solutions are not emerging.

The conception of "knowledge about knowledge" set forth by B.O. Smith in his influential book *Teachers for the Real World* (Smith, 1969), which has had numerous printings, offers a perceptive and pedagogically oriented view of the general problems raised in this part of the report. Smith considers this knowledge about knowledge as a third dimension in addition to knowledge of the subject and knowledge of pedagogy and psychology. He says: "It has only recently been recognized that there is another sort of knowledge that can influence the performance of the teacher: that used in thinking about the subject matter and the logical operations used in manipulating it . . . To handle the subject-matter of instruction in certain difficult situations the teacher does well to understand its elements, its logical dimensions, its uses, its relation to pupil needs, and the degree of its social 'neutrality'. Because teachers do not possess such understanding, they frequently handle the subject-matter of instruction in superficial ways. Consequently, class discussion often suffers from undue vagueness and ambiguity, from unfounded and unchallenged claims, from a failure to develop the significance of the content." (Smith, 1969, p. 125 et seq.). Smith's book does not represent his thinking only, but is a protocol written by an individual about a comprehensive discussion that has taken place in the United States. It was directly introduced into mathematics teacher education through the Teaching Strategies Project of the Georgia Center for the Study of Learning and Teaching Mathematics.

But this relatively high consensus concerning the need for a metaknowledge is not limited to the United States. The concrete expressions and consequences of this general agreement differ very much from one country to another. The second of the aforementioned aspects of the problem under discussion especially requires the formation of a conception of mathematics teaching in schools of general education that relates the content and the methods of the subject matter to the development of the child's personality, as well as to the social function of mathematics.

The metaknowledge thus related to the subject matter that would result from a concretization of both aspects for mathematics teacher education could provide a decisive link between the different competencies of, and demands on, mathematics teachers.

To the greatest part of society the mathematics teacher is the primary source of people's views on and attitudes towards mathematics, and the question of knowledge about mathematics on the teacher's side is therefore not a merely academical question but of great social importance. The following three problem areas make concrete how such knowledge about knowledge could enter into mathematics teacher education (16).

(a) *Methodology and theory of scientific work* (e.g., heuristics, epistemology, etc.): Isolated initiatives in this field can be observed in several countries, e.g., U.S.S.R., France and the United Kingdom.

(b) *History of mathematics*: A whole series of existing programmes for teachers is described in volume 2 (1975) of *Historia Mathematica*. The introduction of such programmes seems to have received growing support, though the usefulness of such historical studies is still heavily disputed, especially where priorities have to be set in mathematics teacher education in view of time constraints.

(c) *Knowledge about the relationships between mathematics and other sciences or school subjects*: Even the treatment of problems of applications requires a minimum knowledge in other areas of experience; it is all the more so for difficult teaching goals such as interdisciplinary teaching or problems of remedial or "mixed-ability teaching". A basic stock of such knowledge is often laid in the second or even third subject taken by students of mathematics, but it has hardly been exploited consciously in teacher education. (Chapter XII is devoted to the interaction between mathematics and other school subjects.)

3.3 Mathematics teacher education and the pedagogy of mathematics

Until the end of the 1960s research and development activities concerning mathematics teaching and mathematics teacher education were traditionally in the hands of mathematicians and mathematics teachers. In the late 1960s and early 1970s, the influence of educational sciences, and especially of empirical social sciences, grew. The pedagogy of mathematics started to emerge gradually as an autonomous discipline, accompanied by a corresponding mediation of the views and methods that have been used in mathematics education and mathematics teacher education. Gradually the social group of those professionally concerned with research, development, and teacher education in the field of mathematics education is developing into an academic group with a social status (17), though it is often the case that "no formal agencies exist to study systematically and explore the didactics side" (Kempa, 1976; cf. also *Westdeutsche Rektorenkonferenz 1975*). In many countries, it has only recently become possible to take a doctor's degree and to qualify for lecturing in the pedagogy of mathematics; in other countries, there is no such possibility. If one considers a key problem in teacher education to be the professionalization of educators, one can draw consequences not only for the content, but also for the organizational and institutional framework of work in the pedagogy of mathematics (compare, for example, Long et al., 1970).

Corresponding to this development, the pedagogy of mathematics concerned itself first with relatively narrow methodological or subject-oriented teaching questions. But at the same time the pedagogical concept of education left out the specific aspects of the subject of instruction and often proved inconsistent with the structure of the subject matter. There has frequently been a poor parallelism between the narrow perspective of the subject of instruction, or of pedagogy of mathematics, and a "pragmatic" surrender of the real teaching process to random and uncontrolled influences on the teacher's behaviour.

As a result of the increased importance of educational issues; of the vast extension of education, frequently accompanied by acute social and socio-psychological teaching problems (classroom management, mixed-ability teaching, etc.); and of the stormy development of the sciences, especially mathematics and its applications in all areas of society; research, development, and university teaching concerned with mathematical education gradually show – hesitatingly and not without contradictions – a completely new silhouette. The most central problems for teacher education are undoubtedly those of mediating between theory and practice, between the subject matter and the social and educational sciences.

More often than not, discussions of the pedagogy of mathematics are dominated by unrelated and, consequently, "shapeless" antagonisms. One of these antagonisms worthy of consideration here, since it is often mentioned as an essential characteristic of the behaviour and goals of primary school teachers as opposed to those of their secondary school colleagues, is the false dichotomy of "child-orientation versus subject-matter-orientation". It is not easy to see what problems this dichotomy conceals. There is very little communication today between primary and secondary school teachers and virtually no exchange of information on their respective situation and problems. New developments, with new demands on the whole school system, have aggravated traditional problems in a way that found expression in the false dichotomy. The traditional conception of the role of subject-oriented pedagogy is being challenged at both school levels. In contrast with the above motto, which sees relationships on quite different planes, these developments deal with all levels of the teaching and learning situation; that is, the level of knowledge (mathematics and its applications), the social level, the level of the socio-psychological relationships, and the level of pedagogical methods and educational behaviour.

The secondary school mathematics teacher has been not only relatively subject-matter-oriented compared to his primary school colleague; his conception of the pedagogical relationship in the classroom has accordingly been different. And this conception corresponded to the social composition of his classes and the structure of the school system. Secondary schools traditionally have been very selective. Only children from affluent sections of society with a high degree of motivation and relatively ambitious educational goals went to grammar schools and similar institutions. In view of the few motivational problems (education as a status symbol) and of the help given by the parents whenever necessary (private lessons, etc.), classroom teaching could centre completely on the content and could produce, correspondingly, pedagogical conceptions holding that knowledge could be transmitted automatically. The goal of teacher education was accordingly to provide the teacher with a good knowledge of his subject matter, in harmony with the belief, as has been said before, that he who had acquired knowledge needed no further abilities to transmit this knowledge. Conversely, the "child-oriented" attitude of primary school teachers not only implies a different organization of the teaching process – i.e., one differing from a teaching form strictly structured by the subject matter – but also other conceptions of the knowledge to be transmitted in mathematics instruction. A "child-oriented" teacher may put more emphasis on the steps leading from the problem to the mathematical subject, from everyday experience to mathematical methods; he may put more weight on the inter-relationships between the different forms of knowledge, e.g., between mathematics and natural sciences. He may also stress drill on elementary skills and is likely to have a conception of knowledge as an intuitive and implicit process rather than an objective explicit structure. Paradoxically, this often leads to a less dynamic content and conception of teaching (18). Where a more sweeping change in the content is enforced by external social developments, there may be a complete split between the content and the methodological-pedagogical levels, as is expressed by the opposition "content-oriented versus child-oriented" (cf. Davidov, 1967).

A constructive and fruitful analysis and discussion of the situation, or of developments in mathematics teaching, should also consider trends aimed at dissolving the differences between these diverging conceptions of the pedagogy of mathematics. We could then analyse more accurately why secondary school teachers view themselves, today more than ten years ago, as members of a united profession rather than as members of the mathematical community, and why secondary school teaching is increasingly drawing on pedagogy and psychology. This would, conversely, facilitate a better assessment of the emergence of the principle of teacher specialization in primary schools, and not only the formalistic exaggerations and reductions of the concomitant curriculum changes, but also the necessary accelerating elements would then become visible.

The present situation of the pedagogy of mathematics is characterized by great conceptual deficiencies. Neither a theory of instruction nor a theory of teaching exists even in outline form, in contrast to the many psychological theories of learning. On the other hand, we are faced with a host of practical problems that can no longer be handled by the conventional inventory of spontaneous principles gained from experience and transmitted only by tradition. Today, even time-honoured pedagogical training methods, such as the "principle of congruence" (19) or the principle of fully integrated transmission of mathematical content and knowledge of the pedagogy of mathematics, are violently disputed.

Today, teacher education and pedagogy of mathematics must meet the demands made by two directions of development: on the one hand, the trend toward a far more accurate and differentiated special knowledge and specialization; on the other hand, the trend toward a more active participation of new groups of people and the inclusion of a growing number of areas of experience in discussion, analysis, and decision making.

This twofold tendency confronts teachers with new demands that are now under discussion in many countries: teachers must maintain their mathematical competency or even extend it; but beyond that, they must acquire the fundamentals of other subjects, in areas where mathematics is applied as well as in pedagogy and psychology, so they can play an active part in the development of the necessary communication and co-operation, and master the many additional problems of justification with which society and parents confront them.

Given this situation, it is becoming ever more important to work out a differentiated and simultaneously practice-oriented methodology of the pedagogy of mathematics. This methodology should be problem-centered and should allow one at the same time to develop fundamental questions in close contact with problems. It ought to be simultaneously eclectic and integrated (Otte, 1974, especially p.127).

In particular, we should consider the systematic development of a differentiated case study methodology, the development of methods for process evaluation instead of mere product evaluation. Such methods are needed especially for the design and interpretation of large-scale experiments and developmental projects in school practice, or for the study of problems raised by the work of dissemination agents, etc. Further, it would be desirable to work out a methodology for discussing problems and goals in the pedagogy of mathematics that would allow the systematic exploration of alternatives and promote the necessary specialization.

The necessary continuous communication among different participants can only be achieved if we depart from the hasty dissemination of isolated recipes or phrases like "the good teacher must . . ." or "teacher education should . . ." that have hitherto been common practice. Today, hasty syntheses, recipes, and evaluations in the absence of suitable evaluation standards gravely hamper a co-ordinated identification and analysis of real problems of mathematics teacher education and of mathematics teaching. Fewer prescriptive and more descriptive statements are desirable, contributing not to arbitrarily chosen and eclectically connected elements of theories, but to efforts as comprehensive as possible aimed at a more accurate view of the different problem levels that are relevant for mathematics teaching, excluding fallacious alternatives.

The hitherto insufficient elaboration and inaccurate identification of problems, the rampant catchwords, the hasty syntheses and judgements – these are all related to the general lack of clarity and stability of the goals of mathematics teaching and mathematics teacher education. It can be observed that the problem of goals is, on the one hand, one of the most controversial and vigorously discussed issues, whereas on the other hand, we lack the conceptual, methodological, and institutional foundations for a well-developed discussion of objectives.

The feeling is widespread that a more complex perspective is needed in the pedagogy of mathematics and that the treatment of its problems calls for more differentiated methods. In the first pages of their book, Griffiths and Howson say (Griffiths and Howson, 1974, p. 4 et seq.): "Our general aim in writing this book can be summed up as an attempt to make the reader more 'aware'. It is not our primary aim to improve his technique as a mathematician, nor as a curriculum developer; but we hope that our book will motivate him to improve his techniques and knowledge, to want to supply hard arguments rather than platitudinous generalities".

NOTES AND REFERENCES

Notes

1. Cf. the NACOME report, 1975, p. 97 et seq.; see also Mies et al., 1975, p. 61 et seq.
2. See e.g. Ahlbrand et al., 1969; Kallos et al., 1975; Bernstein, 1975.
3. Whereas in France, for instance, the curriculum is laid down by the central administration in charge of cultural affairs, in England this decision lies in the competence of the head teacher or of the teachers of mathematics in the school.
4. See, for instance, Fisk et al., 1973; Ryan (ed), 1970; Hannam et al., 1976; Wünsche, 1970.
5. Cf. Mies et al., 1975, ch. 3; Stanford Center for Research and Development in Teaching, 1974; and Abdullina, 1976.
6. Cf. Hollins, 1973, p. 303; the NACOME report, 1975; Orrange et al., 1975, p. 47-48.
7. Cf. the NACOME report, 1975, p. 95.
8. This statement calls for a qualification in the case of U.S.S.R. As the development has not been as rapid as in other countries, the problems of in-service training play no central part in the debate, though a steady attention is being paid to them.
9. For an illustration of such a relation just think of an "inverted U" (as the name suggests). " 'Inverted-U' non-linear relationships are examples of the assumption that student outcome usually results from a combination of teacher behaviors..." (Good et al., 1975, p. 47).
10. See, for instance, the case study by Otte, 1976; also Mies et al., 1975, ch. 6.
11. We do not want to investigate here to what extent the conditions under which science is currently organised may also be detrimental to the "natural relationship" of science to teaching in the perspective of science itself. (cf. P. Blau, 1973, p. 103 et seq.).
12. A pedagogical attitude may even turn into an anti-pedagogical one when the concrete conditions are neglected. In Kotasek's words: "It is significant that the advocates of the unity of research and instruction are the main opponents of a pedagogical approach in university work" (Kotasek, 1972).
13. For the problem of the subject-oriented pedagogical principles see Davidov, 1972.
14. *Numerus clausus:* A system for restricting entry to university study in various fields which depends on grades in the *Abitur.*
15. *Abitur:* General certificate of education obtained at the end of secondary education in the Federal Republic of Germany.
16. In countries where the training programmes include philosophical studies these could partly perform such functions.
17. "Until recently few people would have considered describing themselves as 'mathematical educators' –

they were either teachers, teacher trainers, or mathematicians interested in education" (Griffiths and Howson, 1974).

18. In the history of science and of teaching, the dynamics of development and change have always been linked to relatively great efforts to make knowledge more explicit in the sense of a better way of communication across different *realms of experience;* one forgets all too easily, for instance, the impact of Bourbaki's achievement on development.

19. All such principles naturally include numerous conceptions and are likely to be interpreted in as many ways. In a nutshell, the principle of congruence says: "Teachers are taught as they should teach". (That is, in the subject matter: teachers are offered the content as they will teach it themselves.)

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Chapter VII

A critical analysis of curriculum development in mathematical education

A. G. Howson

1 THE MEANING OF CURRICULUM DEVELOPMENT

1.1 Introduction

Any discussion of curriculum development hinges on our interpretation of the word *curriculum*. This has often been taken to mean the syllabus, whether drawn up by a central organization or within a single school, with the result that curriculum development has been viewed as the production of new syllabuses and texts. Such a restricted outlook must, however, almost certainly lead to failure and disappointment. If curriculum development is to succeed, one must hold a wider view of the curriculum than this and plan changes to take into account all of the factors involved. Neither content nor method can be viewed in isolation, and both can be planned only when aims are clear both for education in general and mathematical education in particular. The best plans will founder unless the examination system is revised to test desired objectives, and to encourage the attainment of educational and mathematical goals rather than to act against them. Curriculum, therefore, must mean more than syllabus – it must include aims, content, methods and assessment procedures. Above all, the part played by the individual teacher must be recognized.

1.2 Pressures that can serve to initiate curriculum development

1.2.1 *Societal and political*

Of all the pressures that initiate curriculum development none is greater than that exerted by society.

This century has seen an enormous growth in the quantity of education provided. In the western countries, the attainment of universal primary education was followed by secondary education for all, and greatly increased provisions for university and other higher education. In the developing countries, the recent growth of educational systems has been even more phenomenal and although primary education is still by no means universal, such systems have now begun to expand their facilities for secondary education sufficiently for them to have to rethink their objectives in this area.

Societies, too, have altered radically and the moves towards egalitarian societies have both affected, and been affected by, changes in educational practice. In particular, education has been seen as a means through which existing systems of values can be changed (or, in some cases, buttressed). Employment demands have altered and some developing countries have realized that they will have to emphasize rural, rather than urban life and developments.

Such changes have far-reaching implications for the curriculum. Secondary education is no longer a training solely for the professional man and future mandarin for whom mathematics is a "mental discipline" and "training in reasoning". Raising the school leaving age by one year should pose not the obvious question "What do we do in the extra year?" but "How do we design a five-year course to replace the four-year one?" The coming of comprehensive schools raises the question of whether or not classes should be selected (streamed) by ability. If classes are to consist of children of widely differing mathematical aptitudes, what are the consequences for teaching methods, and classroom materials?

Such considerations have all led to major activities in the field of curriculum development.

These changes reflect decisions made at a national level. In many countries, however, individual schools and local boards have recently acquired greater freedom in curricular matters and as a result have attempted to meet particular local needs. Socially deprived neighbourhoods, those with large immigrant populations, those linked with a particular industry or type of work, all have their own curriculum problems which demand, and are increasingly receiving, individual solutions.

1.2.2 *Mathematical*

The phenomenal growth in the provision of education has been paralleled by that in mathematics. Not only has nineteenth century mathematics been seen in a different perspective, emphasizing structural considerations, but new branches of the subject have mushroomed. The computer has been responsible for a change in emphasis and mathematics has found a wide variety of new applications as disciplines such as economics and geography have become more and more quantitative.

That such developments should be reflected in curricula is only natural. That it should prove easier to introduce new content in the universities, with their small, highly-qualified teaching force, rather than in the schools is not surprising. Yet changes at a higher level of education impose another pressure on lower levels – the need to prepare students for a new approach to the subject. A change in the curriculum at any level of education is bound to cause problems at the relevant interfaces.

Indeed, many problems at the school/university interface would appear to be caused by the universities having been quicker to respond to mathematical pressures than to social ones, whereas the schools, because of their broader educational aims, have been more concerned to meet society's demands.

1.2.3 *Educational*

The majority of the pressures mentioned in section 1.2.1 were exerted from outside the educational system. Pressures for change do arise however from within education as a result of research, new educational theories, or the pioneering developmental work of individuals. All of these, together with the new possibilities arising from advances in technology (such as the pocket calculator), should stimulate curriculum development. It is unfortunate though that the majority of initiatives arising from such sources have been as restricted as those which attempted to satisfy only mathematical pressures. Content and overall aims have been subjugated to the introduction of "discovery methods", "individualized learning", "a systems approach", "an integrated day", "computer-assisted instruction", etc.

The need to make a coherently conceived response to all the various pressures on the curriculum is clear. How it is to be done is less obvious. It will certainly necessitate a rapprochement between educational researchers and subject specialists. Researchers must see it as a major responsibility that their work should have impact on society and, in particular, on school curricula, and that their results are disseminated more widely and in a more comprehensible manner.

1.2.4 *Change*

"Plus ça change . . ." may well have validity in education; but change itself can also serve to generate yet more changes. The desire to keep abreast of one's national or international neighbours is great. Not every country in the 1960s could have an SMSG (1), but to attend an international gathering with no project on which to report was shame indeed. There were also commercial pressures for change: publishers and authors saw new markets; individuals new challenges.

The desire to change, to act as an innovator, to explore new areas, is strong in most intelligent people. It is natural that many educators welcome the opportunity to deviate from practices that are becoming routine. Innovation is exciting, attracts others' attention to one's work and often brings professional advancement.

1.3 "Units" of curriculum development

So far we have only mentioned one specific project, the highly influential School Mathematics Study Group of the United States. This and similar projects will be considered in more detail later. However, if misconceptions are not to occur, we must emphasize that curriculum development does not solely comprise the efforts of such major bodies.

It is perhaps of value to distinguish between three varying types of curriculum development:

(a) *Large-scale*, i.e. initiatives that are meant to have validity throughout a school system. In countries with a centralized system, the courses developed will be imposed on all schools. In other countries the materials will be prepared in the hope that they will be used widely (e.g. by the SMSG in the United States or the SMP in England). In both cases, the overwhelming majority of those teachers who use the materials will have played no part in their writing, planning or testing.

(b) *Local*, i.e. a small educational system or group of schools within a region which combine – often under the guidance of a full-time "professional" curriculum developer – to produce a new course. A distinguishing feature is that all the schools, and representative teachers from them, will be closely involved in the course design. An example is the Caribbean Mathematics Project.

(c) *Individual*, i.e. a single school or individual teacher is our "unit". Two teachers may co-operate on team-teaching, or an individual may prepare a set of work-cards to cover a particular topic. Dissemination problems vanish.

Clearly, each type of development will present its own problems and offer differing possibilities. These will be discussed in section 3.

1.4 Levels and domains of curriculum development

Curriculum development can take place at all levels of education from kindergarten to university postgraduate and with all types of learner in mind. It is obvious, however, that the processes of change and problems of dissemination will differ considerably according to the age and ability levels of the students and the qualifications (often related) of the teachers concerned.

Thus at a university level where the teachers' mathematical qualifications are high and individual institutions have a high degree of autonomy, development usually takes place in "individual" units. Larger-scale projects rarely arise and when they do (e.g. the (American) Committee on the Undergraduate Program in Mathematics) the outcome is usually a set of guidelines to be interpreted by individuals rather than an exhaustive collection of classroom materials. The greater mathematical expertise of the teacher allows him to respond more quickly to mathematical pressures. (Yet equally, his lack of "educational" qualifications may lead to his being disastrously out of touch with the real learning needs of his students.)

At the other extreme, the primary school teacher is often lacking in mathematical knowledge – indeed, he or she might well have a fear of or hatred for the subject. To effect worthwhile changes in this sector will, therefore, require different techniques, and problems of dissemination may hinder a "large-scale" approach. Again, students of "less than average ability" will often be taught by the less well qualified mathematics teacher, and once more there will be difficulties in ensuring that the intentions of large-scale projects are translated into classroom activities in a truthful and ungarbled form. On the other hand, "local" and individual action will be handicapped by the inability of many teachers to take decisions concerning aims, content and method from a position of mathematical strength.

1.5 Initiation

Pressures for change can, as we have seen, come from many quarters. How changes occur and the biases are built into the innovations will greatly depend upon who is responsible for making the first moves. That SMSG grew out of two conferences of university mathematicians, that the impetus for SMP came from schoolteachers, and that reforms in France were largely caused by outstanding professional mathematicians explains much of their later history, successes and failures. Some projects have arisen because educators not directly involved saw a need for them, others because those concerned in the classroom wished to alter the *status quo*. This difference has often proved to be critical in the dissemination process.

Yet an outstanding and most worrying feature of most projects is that initiation has usually been unilateral in the sense that it was made by mathematicians or by educationalists, and, indeed, that it often concerned mathematics alone, mathematics *qua* mathematics, and not mathematics as a component of a general education.

Decisions concerning subject areas for reform have too often been made by subject-oriented power groups without regard to the country's overall educational aims. Yet who is to make such decisions in a non-centralized educational system? If it is not the subject-specialists, it may well turn out to be the powerful teachers' union which may demonstrate other, but equally undesirable, biases. There is no simple solution.

How too are decisions to be made concerning the allocation of resources, financial and otherwise, which will be needed? In the early 1960s projects were looked on as "experiments" and many included that word in their title. Yet the educational system is not like a chemical laboratory where a failed experiment means merely that certain chemicals and apparatus have been expended. A failed experiment in education means deprived pupils, dispirited teachers and parents who will resist further proposals for change. It is now becoming clearer that not only money is lost when a project fails. There is a need to weigh up the probabilities of success or failure before the decision to allocate resources is made.

The question of what constitutes success is, of course, a difficult one to which we shall return in a later section.

1.6 Goals and objectives

The emphasis laid on the enunciation of goals and objectives differs greatly from country to country. Yet even where these are not explicitly stated, they are to be found implicitly in materials and actions, and an absence of any consensus on objectives will soon become apparent in the materials produced by any group. Who, however, is to be responsible for determining such goals? In the 1960s this was often done by mathematicians, and much of the discontent arising from the reforms of those years arises from this single fact. An inspection of the materials produced in most countries would suggest that mathematicians from outside schools and universities, scientists, industrialists and other employers, had very little say in the fixing of goals. Recently the mathematician's responsibilities to users and employers have been seen much more clearly. Even so, decisions are still being taken within a restricted context. Developers must accept the existing position of mathematics in the overall curriculum and plan how best to use the time allocated to it. Wider considerations concerning the place of mathematics in the general education of the adolescent lie outside their powers. If decisions affecting the mathematical curriculum are to be taken from a general educational standpoint, then it will be necessary to involve general educationalists, employers, politicians and administrators in the planning process. If such participation is to be fruitful, then it will almost certainly have to be locally-based. Precedents do exist, but because of the number of constraints involved at the various interfaces with employers, higher levels of education, etc., changing the balance of the curriculum as a whole is always much more difficult than making changes within a single subject area.

Whether goals and objectives are implicit or explicit it will, as we have seen, still be necessary to revise them from time to time, and for that reason it is essential that they are not presented in a form that can be changed only as a result of a major upheaval. (In this connection, one notes that a "standardized test" is an implicit statement of objectives. Rigid adherence to such tests can, therefore, prove a major obstacle to curriculum development.) Scope too must be given to the individual teacher to adjust the emphasis in what he considers to be in the best interests of his particular students. Yet doing this demands a degree of maturity and professional competence which not all teachers possess. International practice seems to vary between denying the teacher any (official) freedom of action and asking him to bear responsibilities and make decisions when he is ill-equipped to do so. The need *gradually* to foster the professional competence and decision-making capabilities of the teacher is rarely observed.

1.7 The contribution of curriculum theory

Curriculum theory has, like curriculum development, a considerable history. The "objectives" model was described as long ago as 1918, although nowadays we tend to associate it more with the work of Tyler and the Chicago School, which, although considerably more recent, still predated the post-Sputnik wave of curriculum development. At the time when the 1960s' projects were planned, there existed a considerable body of curriculum theory.

In some countries, for example, Sri Lanka (formerly Ceylon), attempts were made to base changes in practice upon this theory. In many countries, however, developers chose to ignore the theorists and in so doing perpetuated the division which has traditionally existed in education between practice and theory.

It is not altogether surprising that mathematicians should suspect the theorists' offerings. They are very much concerned with the setting up of acceptable models and many would object to the validity of, say, the "objectives" model as a description of educational aims and would agree with the view that education as an induction to knowledge (as opposed to education as training and instruction) is successful to the extent that it makes the behavioural outcomes of the students unpredictable. Thus an over-reliance on the "objectives" model will, in their opinion, have a harmful effect on education. Yet there is a profound difference between rejecting a theory because of one's objections to it, and setting it aside through ignorance; and it must be admitted that many active in curriculum development are unaware of the attempts to construct curriculum theories. On the other hand, one is drawn to the conclusion that many theorists seem to be ignorant of the day-to-day problems presented by curriculum development.

Both sides would appear to have much to learn from each other and until they do so the contribution of theory to curriculum development is likely to be relatively minor and not always helpful.

1.8 Approaches to curriculum development

As indicated earlier, approaches to curriculum development reflect the interests and biases of initiators and it is valuable to attempt to classify the various approaches used. Any such classification is, of course, bound to create problems and few projects will be easily labelled (2). Yet the classification does illuminate certain major differences and also indicates how emphases have changed with time. Indeed, often a particular project will be found to have gradually changed its approach. The varying roles of the teacher in the different approaches and the disseminatory problems consequent on this should be noted.

(a) *The "New Math" approach*. An attempt, often university-dominated, to bring the content of school mathematics more in line with current university practice. (SMSG in the United States, the Lichnerowicz Commission in France)

(b) *The "behaviouristic" approach*. An approach which is indifferent to content but which emphasizes a particular learning method, for example, programmed learning, computer-assisted instruction. (IPI Math)

(c) *The "structure of the discipline" approach*. An approach influenced by psychologists such as Piaget, Bruner and Dienes, in which the way of learning is closely linked to the content in an attempt to bring together the structures of thinking with those of mathematics. (CSMP, TEMLAB)

(d) *The formative-oriented approach*. Here mathematics is but one contributor to more general educational aims: the development of the personality, and of cognitive and affective structures. (Madison, SEED)

(e) *The "disciplines-integrating" approach*. Learning is now centred on "real" problems and as such tends to be multidisciplinary and project-oriented. (USMES, MMCP)

2 CURRICULUM DEVELOPMENT AGENCIES

2.1 Curriculum development historically

Curriculum development has a history as long as that of education itself. Significantly, methods used over a thousand years ago are still valid today: the personal search, at home and in other lands, for new books, new studies and new ideas.

Yet by the nineteenth century, education was becoming institutionalized and schoolmasters, if not university teachers, were finding their freedom of action curtailed. In many countries, an examination system had come into being which effectively determined the curriculum that a schoolmaster should follow. An individual teacher ceased to have the power to effect changes. Only when united were teachers able to attack the existing order. The result was the emergence of a new agency for curriculum development, the subject association of teachers. By 1914, there were many countries with mathematical associations and periodicals devoted to mathematics education. Not only did teachers meet nationally to discuss curriculum development, but there were thriving international contacts. An International Commission was established in 1908 to compare methods and syllabuses for mathematical education in various countries and meetings were held to discuss such topics as the mathematics to be taught to students of the physical and natural sciences, the place of rigour in mathematics teaching, and the place of calculus teaching in schools.

The problems look familiar; the context was, however, very different, for the questions referred to small, élitist educational systems with relatively highly qualified staff in which dissemination and other problems did not loom so large.

Alas, the 1914–1918 War brought such international co-operation to an end and left the countries involved in a disastrous economic state, and having lost a generation of prospective innovators. As far as mathematical education was concerned, the years between 1918 and 1960 were not distinguished for innovative enterprise. They did, however, see a marked growth in the provision of education, and a corresponding rise in the membership of the subject associations.

This growth in size had the paradoxical effect that when the need for changes became more widely recognized, the associations were incapable of meeting it. They no longer consisted of enthusiasts drawn together by a common wish to reform, but were rather a cross-section of the more enlightened, but still far from revolutionary, members of the teaching profession. Little impetus for reform, therefore, was to come from the large, established subject associations in the United Kingdom, the Federal Republic of Germany and the United States. The need for a new type of agency arose.

Before describing the new agencies, it is worth noting that the subject associations do still have an important part to play in curriculum development. In countries with a centralized education system, such as Poland and France, the association is a powerful means for exerting influence on the education ministry. In developing countries, the young, small and vigorous association has often been in the forefront of curriculum development work, as in Ghana. Where an older association exists, there is always the possibility of a splinter group of reformers detaching itself and forming a new association. This occurred in the United Kingdom in the 1950s and led to the founding of the Association of Teachers of Mathematics (ATM).

In general, though, associations in non-centralized educational systems have tended to restrict their developmental activities to the "local" and "individual" levels, by providing reformers with space in their various journals and reports, and opportunities to describe their work at annual conferences, and by providing support and backing for small groups of innovators.

There is, however, one enormous drawback to the subject association as an agency for curriculum development: usually it is very badly represented in primary schools. In such schools, the teacher rarely thinks of himself as a subject specialist. He, or more usually she, is called upon to teach most subjects in the curriculum and cannot belong to all the relevant subject associations: most often he will belong to none. It is still possible for subject associations to establish contacts with such teachers through special meetings and publications directed at the primary level. It is, however, considerably more difficult to establish those long-term relationships that will lead to a very marked improvement in the way in which the teacher teaches the subject: again there is a need for a different type of agency.

2.2 "Modern" agencies for curriculum development

2.2.1 *The Project*

It is impossible to identify the first curriculum development project in mathematics, but that honour might go to the late Max Beberman's University of Illinois Committee on School Mathematics (UICSM) which was established in 1952. Certainly, that possessed many of the characteristics of later projects: a group of authors produced materials which were tested in pilot schools and later revised, and it organized courses at which teachers were trained to use the materials. Moreover, it recognized the dangers inherent in the dissemination process by refusing to let its units be used by teachers who had not been so trained.

The UICSM's impact was, however, restricted. It took the Physical Science Study Committee (PSSC), and in its wake the SMSG, to excite general interest in the problems of curriculum development. Moreover, as interest grew in other countries it was usually the PSSC/SMSG model which was followed, even if the proportions had to be scaled down. For at least twelve years, what came to be known as the research-development-dissemination (RDD) model was to prove the most popular way of attempting to effect change.

2.2.2 *Central agencies for curriculum development*

As indicated in 2.2.1, projects were set up in subjects other than mathematics and this led in some countries to the establishment of national bodies for the co-ordination of curriculum development. In other countries, national bodies or a network of centres have been established which are specifically concerned with mathematics. The powers and responsibilities of such institutions differ considerably and their effectiveness as agents for curriculum reform vary likewise. It is instructive, therefore, to consider some of them in more detail and to indicate strengths and weaknesses.

The Schools Council for the Curriculum and Examinations was established in England in 1964. The name indicates one of its strengths, that it is responsible for co-ordinating work not only on the curriculum, but also on how it is to be examined; here it differs from, say, some of the curriculum agencies of the Federal Republic of Germany (e.g. KOSEK). Moreover, it is also responsible for all the curriculum in all types of schools. In theory, therefore, it is capable of effecting changes in the overall balance of the curriculum (see section 1.7) (what happens in practice is another matter, for as we have indicated earlier there are many obstacles to be overcome), and it can more effectively ensure that interface problems do not arise between the various levels in the school sector. The Schools Council is teacher-dominated in the sense that practising teachers are in a majority on all its committees although other interests are also always represented. One great weakness is that it has no official responsibilities for the in-service education of teachers. As a result, the successful dissemination of its work is difficult. Since its projects (like the SMSG and others which are "Foundation" funded) have a limited life-span, this means that frequently the provision of in-service education comes to an end with the project, exactly at that critical time when the project's final materials are being published and major dissemination problems are about to begin. This, however, is a problem that is to some extent disappearing as a result of the trend to encourage "local" rather than "large-scale" projects, and since it has now assumed some responsibility for "after-care".

The Schools Council's work is also restricted in that it is responsible only for educational research concerned explicitly with the curriculum. More "fundamental" research (which must still be seen in a curricular context if it is to be effective) comes under the aegis of another national body.

The IREMs, a network of 25 institutes established in France, represent a very different approach to curriculum development. They are concerned solely with mathematics and have responsibilities for in-service education as well as experimentation in the classroom, the production of films, books (e.g. the *Livres du Problème* written at the Strasbourg IREM) and other teaching materials, and research within the general field of mathematical education at a school level. The work tends to be on-going, but the IREMs, in addition to the drawback of being one-subject institutes, suffer from existing in a centralized system where much of mathematics teaching is dominated by the pressures exerted by higher education and the *baccalauréat*. Officially, too, the work of the IREMs is confined to secondary education.

Yet another mode of working is illustrated by the IOWO, the central institute of the Netherlands for the development of mathematical education of 4–18 year-olds. Initially, in 1971, IOWO inherited a number of duties and projects to supervise. Since then, it has launched a major primary-school project which seeks to replace RDD methods by co-ordinating development and in-service education.

Although IOWO has many similarities with the IREMs, it is essentially distinguished by being a central institute in a non-centralized system, as opposed to being a local institute within a highly-centralized system.

The IOWO could however, only exist in a compact country like the Netherlands where the educational system is relatively small and all schools can be reached by car in two or so hours. Such an agency would be inconceivable in, say, the United States. Yet the advantages of a central agency with access to varied samples of schools, to statistical expertise and to mathematicians and mathematics educators is self-evident. Moreover, if practice and theory are to be united and if developmental work is to be firmly based on more elaborated theories of teaching and learning, then it is likely that centres such as the IOWO and the IREMs will have a major role to play. For in them various expertises are present and the work is concentrated and on-going. The Schools Council, by comparison, is largely an administrative, policy-recommending body which finances developmental work taking place in a number of different institutions. This results in a lack of co-ordination between even those projects devoted to mathematics. In this system, more people are involved in curriculum development, particulary since the membership of committees is constantly changing, but the commitment is often intermittent and/or temporary.

There is, however, the danger that such bodies might overshadow and even discourage the individual innovator. Will they necessarily lead to a greater centralization of education? Some agencies, such as IOWO, attempt to overcome the possible dangers of a state-publishing and text-book writing monopoly by distributing non-copyrighted materials which individual authors are encouraged to incorporate into their books.

The idea of a national centre can, of course, be generalized to an international one. The south-east Asian countries have had such a regional centre (RECSAM) for ten years now. In particular, the centre has a five-year project (SEASAME – Southeast Asia Science and Mathematics Experiment) seeking to design, develop and produce prototype packages of modules which it is hoped will serve as examples of what might be done nationally by the eight member countries. Thus RECSAM is able to optimize what developmental talent exists throughout a vast and rapidly developing region – but only at the cost of adding yet one more dimension to the problem of dissemination.

Clearly, other agencies and institutions will possess different characteristics. Yet those mentioned above suffice to illustrate some of the most important ones which would seem to affect the successful implementation of curriculum development.

2.3 The administration of curriculum development

It is not fashionable in curriculum development circles to speak of the problems of educational administration. Yet the way in which a country's educational system is administered crucially affects curriculum development. Often the division of funds and responsibilities between central and local government can lead to a bewildering array of uncoordinated activities. We have already remarked on the division of responsibilities in England. This situation recurs elsewhere. Thus in the Netherlands there are, in addition to the IOWO, institutes for testing and evaluation, for curriculum development (in general), and for research in education. The teacher training system, as often elsewhere, is fragmented according to religious (and non-religious) divisions. Co-ordination becomes not only a problem of money, but one of politics.

Even when educational administration is centralized within a small system, one often finds that "curriculum development" is a new wing in the ministry separated from the older-established "teacher-training department".

Such a division of duties can have disastrous consequences for either the dissemination of project materials or attempts to replace the RDD model by one which seeks to co-ordinate curriculum development and in-service (and pre-service) teacher-training.

3 STRATEGIES FOR CURRICULUM DEVELOPMENT

3.1 Introduction

The strategies available for curriculum development are many and varied, yet not all will be open to any one group of innovators. Each project, group or teacher will select from those available with the result that few will choose totally similar or altogether different combinations. Any attempted classification can, therefore, be misleading for it may well help emphasize individual elements rather than their interdependence and the mix actually selected in any instance. The reader is also asked to bear in mind how much the strategies of development are determined by the abilities of the teachers (see section 1.4) and by the degree of their involvement in the process (see section 4).

3.2 Projects: the research/development/dissemination model

A most puzzling aspect of the RDD model is its name, which suggests that a period of "research" is followed by one of "development". This is not how the majority of so-called projects have operated.

There are instances where a project has resulted from a desire to implement educational research findings, or has commenced its activities by instigating research. Usually, however, the connection between research and development has been of a different nature. The developer has usually begun with fairly clear objectives and has sought the aid of research only when particular problems have arisen. Even then he is likely to have set up experimental work of an *ad hoc* nature rather than to have sought a solution in published research findings.

Some people may, however, associate the "research" with the feasibility studies which often precede the launching of projects. One would have more faith in such studies as "research", if it were ever discovered that the proposals were not feasible. Yet even if its name is misleading, a feasibility study does have value. As indicated earlier (section 1.5), decisions concerning the funding and initiations of projects should be taken from as informed a position as possible. The idea of employing an individual or small team to investigate possibilities, to identify relevant experiments and possible contributors to a major project, to determine provisional objectives and to suggest timetables and budgets is a good one.

3.2.1 *Project teams*

The most crucial appointment as far as any project is concerned is that of the Director. It is not sufficient for him to be competent academically – this is necessary, but equally essential is the ability to guide and lead a team; to make sensible appointments; to manage business and financial affairs; to deal with publishers, the press and the educational establishment; and last, but not least, to strike up an easy relationship with, and be accepted by, teachers in the project schools.

The composition of writing teams has varied considerably. Originally it was common to have a small team consisting of full-time workers. These were recruited from different backgrounds: in the United Kingdom such teams usually contained a preponderance of teachers, whereas in the United States the balance was more even or was weighted towards the university academic. The dangers of placing responsibility for writing in the hands of university mathematicians are exemplified in the work of many projects whose materials have long since been cast aside as impracticable and, indeed, it could be argued that however else the team is constituted, it should contain practising schoolteachers with the right to veto any materials they themselves find unteachable. Yet teams which lack professional mathematical support may produce material lacking in coherence, purpose and balance. Indeed the achievement of "balance" is notoriously difficult – if only for the fact that one man's balance is another's bias. To have all interests represented on a writing team is difficult – particularly if the team consists of full-time writers. There is, therefore, a great need to ensure that scientists and other users, industrialists, general educators and teacher-trainers, not only have an opportunity to comment on proposed plans, but are actively encouraged to do so and are involved in the decision-making processes. This is occasionally attempted through the appointment of consultative committees: a device which continually promises more than it achieves.

One of the most successful of all projects – judged, *faute de mieux*, by degree of acceptance and longevity – is the School Mathematics Project of England. This is noteworthy for the fact that not only have all its writing groups consisted almost entirely of practising teachers, but it has consistently used part-time rather than full-time writers. Authors have written in their free time or have been given relief from *some* of their teaching duties to study and write. As a result, they have retained day-to-day contact with the classroom and have been able to try out their ideas there before committing them to paper.

The use of practising teachers as writers avoids some obvious traps. Nevertheless, the teachers who participate in writing teams are far from typical. As a result, they too can set unrealistic goals for their less advanced colleagues – indeed, the wish to appear abreast of recent developments and to be upholders, and advancers, of academic standards can still lead to over-ambitious schemes which are difficult if not impossible to implement.

There are other advantages in using part-time authors. One is that the problem of finding suitable employment for team members at the end of a project's life is substantially eased. In an expanding educational system, work with a project can be a stepping-stone to higher things: this is not necessarily true in a contracting system. Difficulties of obtaining suitable employment can lead to a gradual eroding of project teams as the work reaches its final and crucial stages: team members may put career prospects ahead of loyalty to the project. Far better then to use authors who remain in post and are not faced with re-employment problems. They, too, are able to leave a team in a discreet fashion if, as may well happen, they find that they lack the ability to write creatively, to function as a member of a group, or to withstand criticism. Unlike the full-time writer, they do not remain, seeing their contract out and contributing little to the project's work.

Above we mentioned the need for the writing-team to consult widely amongst scientists and users. It is now becoming increasingly recognized (witness the recent ICMI Regional Conference in Nairobi) that the linguist too has an important contribution to make. In many countries, the cry has been heard that the "language used in the texts" has been pitched at the wrong level. Too often mathematicians have responded to this by having yet another go at the task – not by seeking professional assistance from those concerned with language teaching. Even deeper than the question of what "language" is used in texts and other materials, is the problem of the interaction between the concurrent growth of linguistic and mathematical skills – a neglected area of educational research.

3.2.2 *Trials*

It has been customary for RDD projects to produce draft material to be tried out in a number of "pilot" schools. This has then been revised and experimented with on a larger scale, before a final version was written and published. Often an "evaluator" or group of evaluators has been added to the project team to oversee the trials in the schools, to collect data and criticisms for the writing team, and to monitor the general acceptance of the materials. Such an arrangement brings benefits, yet if it is to be offered as an alternative to close contact between teachers in the trial schools and the writers, then much is lost. Finding out what happens in the classroom, what teachers really think, and how pupils actually respond, cannot be accomplished through questionnaires. The logic of this will almost invariably lead to very small groups of trial schools, carefully selected, closely monitored and with teachers who feel (rightly) that they are at the centre of the project, members of the enlarged team, rather than superior guinea pigs. If this is to be the case, then schools and teachers must participate at their own request and not be delegated to do so by administrators.

There is a danger that with the first appearance of draft materials and in the excitement of testing them in schools, the need to obtain advice and criticism from *outside* the classroom is forgotten. Too often one concentrates on whether the material is teachable, and how it may be made more accessible to teachers and learners. The opportunity to rethink goals and objectives in the light of classroom experience is less frequently taken. Often the project's timetable forbids this: if so the loss is crippling. There is also the need to seek advice and criticism from employers and teachers in other sectors of education who will eventually be affected by changes. It is easy to reach agreement on objectives; only when the materials are subject to scrutiny will it be seen whether that agreement extends from intent to implementation.

How long trials should extend is a matter for debate. Often the effect of repeated revisions is that a high polish is applied to those parts of the materials that are "easy" to teach, while a succession of attempts is made to solve the more difficult teaching problems with the result that the final, published version of some of the trickier parts is still largely untested. This is a problem which presents real difficulties. The solution would not seem to lie in an extended trials period, but in a more flexible approach to the testing and writing of those sections which it is anticipated (indeed, known) will cause difficulties – perhaps by the simultaneous testing of alternative approaches. However, no trials in a relatively small group of committed and closely-tended schools will reveal the problems, misinterpretations and misunderstandings which will arise when materials are widely disseminated. That adjustments can be made at that stage is a far more important need. Unfortunately, by that time many projects have become a part of history, or have packaged their wares in such a way that even minor readjustments of content and emphasis are impossible.

This latter is, of course, a publishing problem. The role of publishers in curriculum development will be discussed later (section 3.4.1). Here it suffices to emphasize the need for project teams to work closely with their prospective publishers throughout the whole writing period. They have more than knowledge of economical methods and styles of production to contribute. Good publishers are likely to be able to suggest alternative presentations of materials and will be able to offer sound advice based on the links they will have established with teachers throughout the country.

3.2.3 *Disseminatory strategies and their effects*

Dissemination is, of course, vitally influenced by whether or not the educational system is centralized (overtly, as in, say, the German Democratic Republic, or in a more hidden form, say, Scotland). The different strategies available have been discussed elsewhere (3), but for our purpose it is useful to identify three that have been extensively used with large-scale projects. It must be noted, however, that the strategies are not usually employed singly, but as a blend.

(a) The *power-coercive* strategy is characterized by fiats such as "from 1st October 19- - all schools shall";

(b) The *pressure-coercive* strategy employs arguments such as "if you cared for your children . . . . , if you were up-to-date, you would be using . . . .". Less emotive and more tangible methods of "encouraging" reform are having a higher pass-rate in the "modern" examinations and paying higher salaries to those teachers who teach the new syllabus;

(c) The *rational-empirical* strategy which seeks to explain to teachers the mathematical benefits which would accrue from a change of curriculum. This strategy is often connected with evaluatory exercises.

Generally speaking, one finds that projects whose work is disseminated throughout a national or local educational system tend to use the two coercive strategies, whilst the "large scale" project working in a non-centralized system makes greater use of "rational empirical" arguments.

The type of strategy adopted is of enormous importance for it serves to define the teacher's role in the eyes of the administrator of education: whether he is to be treated as a servant of the system, a semi-autonomous being to be manipulated or urged along, or a professional capable of making sensible decisions. Not only will the strategy used affect a teacher's initial reaction to a project and its materials, but it also influences the teacher's sense of commitment and the way in which he will later react to any difficulties arising from the materials' use in the classroom.

Although the use of coercive methods is most obvious in centralized systems, they are also to be found in those systems in which teachers have freedom of action. Decisions have often to be made with future career prospects in mind and under the influence of pressures exerted by headmasters, inspectors, and local advisers.

3.3 An alternative model

The failure of many projects to achieve the goals they sought and, indeed, a growing awareness of the difficulty of *radically* affecting the way in which mathematics is taught in the majority of classrooms have drawn attention to the limitations of the RDD model. The dissemination problem (in which we include implementation) has not been solved. Frequently aims have not been adequately communicated to teachers, or the understanding needed to achieve them has been lacking. Pupils have received a "garbled" message. Attempts to produce "teacher-free" material have foundered simply because in any classroom the influence of the teacher exceeds that of any materials. One cannot impose a curriculum on a teacher: it is he who will impose himself, his strengths and weaknesses on what is offered. Once this is accepted, the logical consequence is that curriculum development must *start* from the teacher. He must be helped to overcome weaknesses and exploit his strengths, and a structure must be supplied within which he can develop as a professional. The dissemination process is turned upside-down and a new approach introduced, the re-educative strategy, combining in-service education and curriculum development.

This is based on the active participation of teachers in the preparation of materials for their classes. It is a method that is being adopted in many parts of the world and has been exploited recently, for example, in the Caribbean (CMP) and in Britain (e.g. MMCP, SMILE (the Inner London Education Authority Project), the Kent Project, the Fife Project). It is curriculum development in the "local" mode.

Merely bringing teachers together and inviting them to write their own materials is, however, likely to be as unrewarding a strategy for curriculum development as is the RDD model: considerable professional and administrative support will still be required. Indeed it is likely that this method will prove more expensive to mount than the RDD model. Yet its aims and potential are greater, for it seeks not only to change the curriculum, but the teacher also.

Clearly, a small group of teachers, if it is to combine active learning with the production of materials, will not be able to produce a complete school course. Thus projects using this developmental strategy have tended to produce "modular" courses with each of a number of small groups contributing units. There are still some dissemination problems, therefore, as teachers are asked to absorb units produced elsewhere into their courses. Nevertheless, the experience of actually producing materials can bring knowledge and self-confidence that facilitates the acceptance of the materials of others and can even lead a teacher to become independent of them.

Organizing curriculum development in this mode – attractive though it may seem – does not allow us to sidestep some of the crucial questions raised earlier. We must still ask "Who initiates such developments?", "How are the writing groups to be formed?", "Who is to determine the subject-matter and aims?", "Who is to determine the style of the materials?", "What resources, financial and other, are required and how are they to be supplied?"

Again, the materials produced by such groups are unlikely – unless strong professional backing and support is supplied – to be comparable in quality with those produced by the highly-selective teams assembled by RDD projects. Is local material then to be subject to "expert" editing? How are the materials produced by different groups to be co-ordinated and disseminated? Will such "home-spun" materials prove as generally acceptable to pupils, parents and teachers as "official" commercially-produced materials with their "authoritative" format?

Only time and further experience will yield the answers to these and other questions. However, already the signs indicate that by itself this "locally-directed" style of curriculum development will be no more successful on a national scale than the "large-scale" projects were. The way ahead, as so often is the case, would seem to be a mix of styles. The large-scale project with its professional expertise and advisory network would be responsible for designing a curriculum to meet national aims and objectives and for supplying basic materials. However, "local" groups would be established to study the nationally-produced curriculum in the light of local needs, to adjust it accordingly and to produce necessary supplementary materials. Thus each school could be drawn into the decision-making process and the professional status of its teachers correspondingly enhanced. Even here though there will be difficulties if the national curriculum is conceived in terms of a "global" deductive system, thus, for example, it is difficult to envisage how the closely-knit curriculum developed in Belgium by Papy could admit meaningful local adaptation.

3.4 Textbooks and other materials

3.4.1 *The role of the textbook and its authors*

It has been said that the textbook is the "hidden curriculum" and there is no doubt that for many years a "change in curriculum" and a "change in the textbook" were regarded as synonymous. Even though the influence of the textbook has recently been challenged, it is still extremely strong.

When speaking of the textbook the need arises to distinguish between several very different situations, which can be classified according to the amount of choice the individual teacher has. In some countries there is no, or an extremely limited, choice (for example, Poland), in others the teacher selects from a wide choice of approved texts (Canada), in others approved texts are supplied free, whereas unauthorized ones if they are used must be paid for by the pupils (Federal Republic of Germany), others give complete choice to the teacher (England). The way in which textbooks are produced also differs. Some centralized systems produce their own texts, others lay down the syllabus to be covered and select by competition the one or two texts to be authorized. Traditionally, in countries such as the United States and Britain texts were written by individuals or pairs of authors. Now, to meet the challenge of project-based materials, it is becoming more common for groups of authors to combine on a series – often brought together for the purpose by commercial publishers.

Group authorship enables books to be produced more quickly and means that a greater pool of experience, examples and creative ideas is available than can be offered by an individual (who may, however, offer greater uniformity and consistency of approach, e.g. Papy).

In certain counties (e.g. the Federal Republic of Germany), curriculum development has been effected through group authorship (built upon educational research and trials) rather than projects, with the publisher often making the first moves. Too frequently, however, textbook writing has been based on market research rather than the educational variety. In a country such as the United States publishing is a highly lucrative and competitive business and providing the mix that teachers will like can be very rewarding financially. By such means publishers exert a powerful – but usually conservative – influence on curriculum development.

Such "mixes" will often incorporate ideas derived from the work of projects – though frequently it is the more meretricious innovations that are retained. In so doing, they illustrate one of the secondary effects of projects. Another is that members of writing teams later, individually or as a group, become highly successful authors of commercially produced texts. Thus SMSG could claim to have trained many successful textbook writers, while in Sweden the successful "Hey, mathematics" series is "descended" from the IMU Project.

3.4.2 *Printed alternatives to the textbook*

Attempts to provide individualized instruction (as opposed to programmed texts which encourage everyone to run the same course, but at their own speed) and learning systems which can be used with mixed-ability groups, have resulted in significant developments in printed materials other than books. Projects such as the IMU have attempted, by breaking texts down into smaller units offered at varying levels of difficulty, to provide a flexible learning system suited to children of widely different abilities. The results have not been an unqualified success. Such systems are expensive both financially and in the administrative demands they make on teachers. A simpler, more economical alternative is the use of work cards. These have been widely used in many countries, but there are indications that such systems are inappropriate for brighter children, since they can spend too much time on work which to them is trivial, and on the collection, checking and return of cards. The former complaint can, of course, be levelled at much traditional teaching – it is only thrown into high relief by the emphasis on individualization.

The idea of a bank of materials, booklets, work cards, expendable workbooks, and non-printed aids, has also been exploited. Increased access to sophisticated reprographic facilities – small litho machines, heat duplicators and electronic stencil makers – has enabled teachers to match materials more closely to their needs. It has encouraged "individual" and "local" developments.

Yet despite these innovations, publishers still assert that most teachers prefer to purchase prepackaged material which they can transplant immediately into the classroom. Topic books and booklets, although beloved by innovators and reviewers, do not sell well.

It is also still the case that teachers are more easily influenced by pupils' materials than by teachers' guides. Pupils' materials, if they are to be successfully used must be accompanied by teachers' guides which go beyond merely supplying the answers. Experience has shown, however, that attempts to effect change by means of teachers' guides alone have rarely been successful.

3.4.3 *Other media*

The reforms of the 1960s were accompanied by a vast growth in the technological aids available to the teacher: film loops, slides, cassette tapes, overhead projectors, teaching machines, television and, overshadowing all, the computer. Great things were hoped, but much less has been achieved: print still reigns supreme. The reasons for this are varied. Cost is, of course, a major item, as is simplicity of use. Other reasons are comparatively temporary; the disappointing standard of software, the non-compatibility of hardware, the lack of teaching skill in the media. More important still, though, is the fact that print – with the more sophisticated methods of illustration now available to us – still has many over-riding virtues so far as the communication of mathematics is concerned. It allows the student to pace himself and to refer back with ease. The other media have much to offer, particularly in the field of teacher training, and mathematical education would be poorer without them, yet it is unlikely that they will ever have more than an ancillary role to play.

3.5 Evaluation

The need critically to inspect and evaluate developmental work is inherent in most innovators and has a history going back to the Book of Genesis. Recently, methods of evaluation, as described in Chapter VIII, have beome more technical. Here we are interested not so much in the methods used, but in the framework within which evaluation is carried out.

We have already referred (section 3.2.2) to the formative evaluation carried out by projects in their trial schools. There are, however, many other types of evaluation which are possible: thus, for example, we can ask how successful the project has been in communicating its objectives to teachers, how the results obtained using the project's materials differ from those obtained using other approaches, what effect the project has had on participating teachers or on individual pupils.

There has certainly been no shortage of "comparative" evaluation if one is prepared to use that term to describe criticisms by university mathematicians, scientists, employers and parents of the shortcomings of "modern mathematics" *vis-a-vis* "traditional". However, if one insists on a less emotional, more scientific, approach to evaluation then one finds less to report. The two major evaluatory exercises have been the National Longitudinal Study of Mathematical Abilities (NLSMA) in the United States and that associated with IMU in Sweden, but there have also been a number of smaller attempts in, for example, the U.S.S.R., Scotland, the Caribbean, and Botswana, Lesotho and Swaziland. These examples show the technical difficulties of evaluation and the political and social problems which beset it.

Indeed, there is a danger that because of these difficulties evaluators will concentrate on technical measurement problems, with a probable over-emphasis on cognitive goals: "behavioural objectives" and "criteria for mastery" become key phrases and there is talk of "minimum competencies" which may in time become maximum expectations. More basic questions are neglected.

For what purposes is evaluation to be carried out and according to what criteria? It can hardly be used to judge "success", for any educational innovation launched with enthusiasm by a non-charlatan is guaranteed to improve certain parts of education, for certain children, in the view of certain onlookers. Equally, we can state that any innovation can be justifiably criticized if the grounds are chosen with care. As NLSMA results demonstrated, such questions as "Which textbook is better?" are not appropriate: they are ruled out of order by the multivariate nature of mathematics achievement.

Again, comparisons are made impossible because of extra-mathematical changes. Thus an apparent decline in mathematical standards noticed at Scottish universities may be due to the introduction of new mathematics courses or to other causes such as the introduction of comprehensive education with its accompanying traumas, or, say, to a lessening emphasis on accuracy within society at large and in other areas of the school curriculum manifested, for example, in the greater emphasis laid on creative writing, at the expense of grammar and spelling.

The data provided by evaluators – even if they appear in an objective form – must, therefore, be interpreted subjectively. But who is to carry out the interpretation and what action will result?

Indeed, it is perhaps useful here to distinguish between the "evaluators" who measure, test and supply data, and the "effective evaluators" who take decisions based on such data and on personal experience.

In a centralized system, it may well be the inspectors who are the effective evaluators with the system reacting to their findings. In a non-centralized system, it is more likely to be individual schools and teachers guided by advisers or inspectors. They will tend to evaluate a course on how easy it is to use and teach, and on how their pupils fare when they take public examinations, apply for a job, or move on to higher education. They will also make judgements on financial grounds and rule out possible alternatives because of cost. It is their evaluation that has led to the demise of some projects and the success of others.

The implications of this would appear to be that increased emphasis should be placed on the school and the teacher as decision takers. Guidance will have to be given on how they can best carry out this task and national efforts will have to concentrate on the provision of data that will enable more soundly-based decisions to be made.

Whether large-scale operations of the NLSMA variety are cost-effective is doubted by many curriculum developers. Thus, in England, the Schools Council has not mounted any comparable exercise in the field of mathematics, an expensive evaluation of the teaching of French in primary schools having merely fuelled a somewhat undisciplined debate.

3.6 "Public" examinations

The examinations taken by a pupil are an intrinsic part of the curriculum and must accordingly reflect aims, content and method. Ensuring that they adjust to changes in content is relatively easy. Unfortunately, they do not always reflect changes in aims and method. The result of this can be to nullify or distort the developers' intentions. Thus, for example, examination questions on changing number bases or dividing in the binary scale will lead teachers to drill pupils in these techniques: something far from the minds of those who introduced different number bases into the curriculum so as to reinforce children's understanding of the denary system. What was intended as a means, a catalyst, is turned by unthinking examiners and teachers into an end. The dangers are self-evident. Our examination systems in the past have been geared to a largely different conception of mathematics teaching. Once this conception changes, the examination system must change also, otherwise those teachers who wish to continue teaching as they always did will find that they have an immensely strong ally.

We have so far spoken of the examination responding to change. Traditionally though, the public examination has also been used to bring about innovation. Reformers have struggled to have syllabuses amended and all teachers have then had to respond to these changes. It is by exploiting this possibility and the greater importance attached to public examinations in their countries that, say, the educators in Germany and Britain have in the past hundred or so years kept school syllabuses more fluid and up-to-date (e.g. by the introduction of the calculus early this century) than their counterparts in the United States. Recently, there have been instances where examinations have been used as an instrument for the introduction of changes other than those in content. Thus, for example, in Sri Lanka it was hoped to effect changes in teaching method by means of new examination procedures: nevertheless, the public examination remains a crude instrument for effecting change.

3.6.1 *The design of new forms of examination*

Recent developments both in the technical field of assessment and in the more general area of curriculum development have led to the design of new types of examination. Technical matters such as the use of multiple-choice, short answer questions do not specifically concern us here. More relevant are the alternatives now offered to the traditional written examination paper. The growth in the use of projects, course assignments, continuous assessment, etc. has been remarkable and has extended dramatically the range over which one can examine – the qualities, the understanding, the originality of students.

Yet such methods demand more from examiners, particularly time. They also confer on teachers important roles in the examination system. They can now become involved not only in the evaluation of the course they give, but in the assessment of their pupils. Thus, in both the Federal Republic of Germany and Britain opportunities exist for teachers to determine parts (or all) of the syllabus which they wish to teach and to design examination papers (or alternative assessment procedures) which are then externally moderated.

4 CURRICULUM DEVELOPMENT, THE TEACHER AND IN-SERVICE EDUCATION

4.1 The teacher and curriculum development

The outstanding fact to emerge from twenty years of frantic curriculum development is the crucial role of the teacher. No matter how outstanding the project's team or materials, the success of its work will ultimately hinge on the receptiveness and adaptability of the classroom teacher.

The results of this are far-reaching. Developments have to be geared to the teachers available; this may be a dispiriting prospect in some countries but it is the only realistic one. Pupils may well be ready to receive new material, but if their teachers are not then disaster threatens. The in-service training of teachers must come before – or at least be conducted simultaneously with – the introduction of new curricula.

Pre-service training can be adjusted to meet the new demands which teachers must face, but this is no answer by itself; indeed, it will only result in considerable stress for the probationary teacher who will have his ideals shattered by what he finds actually happening in the school. Moreover, in many of the western countries, a falling birthrate has led to contracting educational systems in which there are fewer new entrants to the teaching profession each year. The majority of those who will be teaching in the year 2000 AD are already in post. Teacher-training in these countries will have to be concentrated, therefore, on the in-service level. The ill-qualified teacher must be given opportunities to obtain further understanding both of mathematics and the problems of teaching it; those who "find it best to do what they can do – and have had practice in doing" must be given the knowledge and confidence to reassess and renew their work. This will only be achieved through the use of carefully planned incentives. The professional innovator's incentives are there for all to see. It is less obvious why the individual teacher should stir from his rut. He, unlike the professional, has much to lose and little to gain by change.

4.2 The autonomy of the teacher

The degree of freedom which is theoretically granted to the teacher varies considerably from country to country. In some countries, text books and syllabuses are rigidly prescribed, and the teacher has little or no part to play in either their revision or in the assessment of his pupils. Elsewhere there is a greater freedom. In any event, the individual teacher once inside his own classroom can exercise a considerable degree of autonomy. The "non-autonomous" system does not prevent individualism, it simply does not actively encourage it.

That schools and teachers should possess autonomy would seem good – for philosophical and political reasons as well as psychological and sociological ones. Autonomy enables the teacher to initiate change and to participate more fully in developmental work: it is an important factor in any discussion on curriculum development.

Yet autonomy must be earned; there will probably have to be some limitations related to the teacher's own educational standard. Again the degree of autonomy granted is likely to be governed considerably by the customs, traditions and conventions of the society in which the teacher is working. To grant "autonomy" to a primary school teacher in, say, Northern Nigeria, would be meaningless both in an educational and a social context. Yet despite these misgivings and constraints, the pressures inside education are such that it would seem desirable to stress the need for autonomy – or a controlled movement towards it – rather than its limitations.

These limitations are, however, vital factors in curriculum development. In many countries, attempts to encourage autonomy and to involve teachers in decision making and curriculum construction have foundered because too much has been asked of teachers and too little assistance been given. If we wish them to construct new curricula, then primary school teachers and other non-specialist teachers of mathematics will need considerably more help than teachers' guides which offer suggestions but no general framework, or occasional short courses to be fitted in alongside similar courses on teaching the mother tongue, a foreign language, crafts, science and so on. It is essential that teachers are not asked to shoulder more responsibility than they can bear.

Where autonomy exists, the innovator is free to experiment and to initiate. He will be the teacher who is drawn into the activities of the various projects, attends courses and conferences, obtains sabbatical leave, and sits or is represented on the committees of central developmental agencies. As a result, one hopes he will become a better teacher. What will almost certainly happen, however, is that the gulf which separates him from the "autonomous conservator" – the man who uses his freedom to do nothing – will widen. The range of teaching to be found in the schools extends even further.

How are these problems to be overcome? Is one to seek greater standardization and uniformity by limiting autonomy, or is one to retain autonomy and seek instead to instil a greater feeling of professionalism amongst teachers? The former alternative is certainly cheaper, but its potential rewards are far smaller. If teachers, though, are to act as professionals and see it as a duty to keep abreast of developments in their subject, just as a doctor will keep himself informed about new medicines and techniques, or a solicitor about changes in the law, then they must be treated as professionals. Time must be provided for them to study and to plan, and their status in the community, both financial and social, must be related to the responsibilities they bear. This will not be accomplished overnight, and will certainly necessitate a detailed study of incentives, accreditation schemes, etc.

4.3 In-service education

4.3.1 *The teacher of mathematics*

The curriculum developers of the early 1960s quickly recognized the need to provide in-service education for the teachers who were to use the new materials. The methods used varied from country to country and included short courses and conferences (notably the NSF "institutes" in the United States, correspondence courses (U.S.S.R.), teacher's "guides" and "handbooks", films (for example, those of the Madison Project) and television.

Although such methods were effective for particular types of teachers, for example, the well-qualified grammar (academic) school teachers who adopted SMP books in the project's early days, they were quite inadequate for teachers in primary and comprehensive schools who, as it happened, were being asked to make even greater changes to their teaching practice.

In the United Kingdom, the Nuffield Mathematics Project sought to meet this challenge through the establishment of centres at which teachers could discuss mathematical and pedagogical problems and exchange classroom teaching experiences. Eventually several hundred centres devoted to curriculum development were established throughout the United Kingdom. Similar centres are to be found in many other countries; in particular, an extensive system exists in the U.S.S.R.

Although the value of such centres is immense, they do not in themselves solve all the problems of in-service education. Their effectiveness is governed by the expertise on which they can call and the ease with which teachers can attend them. It is difficult to encourage and make provision for regular teacher attendance in towns where it is only a short journey to the nearest centre: in rural areas transportation problems can be overwhelming.

Perhaps the answer to this problem, and to a great extent that of the autonomous conservator, is to be found in school-based in-service education. Here, work can either be directed by visiting staff from a local centre or it can be a co-operative effort led (in those countries where such a post exists) by the Head (or Chairman) of the Department of Mathematics.

This would seem to be a most crucial appointment. That someone in each school should have the duty of overseeing and encouraging the professional development of all those involved in the teaching of mathematics would seem most desirable. The post would not be an easy one to hold and if carried out seriously would be no sinecure. Yet it should provide in-service education with a framework enabling ideas to pass backwards and forwards from central organizations, through local centres and into individual schools. A consequence would be the need to provide a considerable amount of training and guidance for those assuming head of department duties.

In many countries there is no head of department, all mathematics teachers in a school being ranked equal and on the same salary scale. The result of this is often to present " interface" problems within a single school, as teachers go their separate ways with few attempts at co-ordination. It is interesting that in, say, the Federal Republic of Germany there is now a move towards the appointment of departmental heads in schools. This may well become a trend in other countries.

Other key personnel in the in-service training programme will be the inspectors and/or local advisers.

The role of the inspectorate has gradually changed in many countries as the teacher has attained greater autonomy. He is, often, no longer the man who determines that statutes are being observed and who holds the key to promotion, but rather an adviser, a transmitter of ideas. The only danger arises when these two roles – the old and the new – become intertwined. When the man who transmits ideas also has an influence on career prospects, the teacher is tempted to accept them whether or not they are appropriate.

Finally, if their teaching is to reflect society's needs then teachers must be aware of what is happening outside the educational system. The "proletarianization" of the curriculum of China was accompanied by extensive and rigorous re-education in factories and farms for the "bourgeois intellectual" teachers. This represents an extreme view on in-service education. A more generally acceptable approach is that which encourages teachers to form links with industry and commerce and to spend short times working and learning within their institutions.

4.3.2 *The "in-service" education of others*

Not only the teachers of new material need to understand the reasons underlying its introduction, to know of its aims and of its possible consequences. This message was grasped by many of the early projects, but the weight that had to be attached to it, and the amount of effort that needed to be devoted to it were not fully appreciated. Indeed, it may well be that with greater consultation with teachers of other subjects, teachers at other levels, parents, educational administrators, employers and others *prior* to the launching of developmental work, the need for later explanations will be reduced.

Whether or not this will prove to be the case, there is no denying that, in general, mathematical educators have failed (or in many cases, have not attempted) to explain their aims and methods sufficiently to others. Time spent, say, in explaining new notation to scientists would not only have eased relations with them, but would also have caused us to consider more deeply exactly whether or not it was required and, if so, for what purposes. Scientists find that students do not possess the old techniques, but are they aware of new ones which can replace them? Are university mathematicians and employers in general aware of the conditions under which teachers operate and of the reasons which have led to the taking of decisions which may affect the performance and understanding of undergraduates and craft apprentices?

These and many similar questions, which all resolve themselves into a consideration of improved communications, only emphasize the importance of mathematics in and to society. Not only does society instigate curriculum development, it also must be informed of the way in which developments are proceeding.

4.4 In-service education and curriculum development

It will be clear from the preceding sections that it is now becoming generally recognized that curriculum development and in-service education cannot any longer be viewed as two related, but chronologically distinct, tasks.

In some countries, it may still be thought necessary to persist with the RDD model and to supply teachers not with *in-service education*, but with *in-service training* in the use of the project's materials. Past experience suggests, however, that this course of action is unlikely to be very successful. If new materials are to be handled with understanding, then training is insufficient – one can train teachers to handle a new learning system, yet to cope with the difficulties which arise in its use, the teachers must be re-educated. The number of possibilities that can arise within a classroom are too great to be foreseen and forestalled.

Yet if in-service education and curriculum development are to become wholly integrated then this will require careful planning and oversight. Some of the problems have already been touched on in section 3.3, where it was urged that "local" developments should be provided with, and should build on, a framework of materials developed by professionals.

The problems of the autonomous conservator, and the would-be innovator who is inhibited by tradition – his own schooling and the knowledge of what has succeeded, and been appropriate, in the past – will not disappear overnight. Indeed, it is in recognizing this fact that one takes a step towards solving the problem. Changes in education take years to accomplish and to diffuse throughout the system. That this should have been ignored by projects which have attempted to mount educational "coups" within extremely limited time spans is surprising and accounts for much of the disappointment that has been felt. A week may be a long time in politics, but a decade is a short period in education. It may see many superficial changes in names and styles, but the personnel concerned and their attitudes are less easily changed. Successful innovation in education is more likely to be modelled by analytic continuation than by catastrophe theory. Too sudden and drastic changes impose overmuch stress on teachers and pupils alike. Curriculum development should be a gradual process: continuous, not a step function.

Reaching all teachers will therefore take time. Yet in several countries, integrated "local" developments have already succeeded in producing a cadre of teachers who are committed to developing a, and to a developing, curriculum. Many teachers will respond in a conservative way at first but if they learn to view curriculum development as an on-going rather than a one-off process, then further progress can be made later. Moreover, it will be made from a position of strength and confidence rather than one of despair. Clutching at straws in the way of simplistic solutions to extremely complex problems, whether the solutions be individualization of instruction, modular scheduling, maths labs, open schools or even a change of syllabus, will one hopes become a thing of the past.

In-service education will not achieve these objectives if it is interpreted in too narrow a mathematical sense. It must also provide the opportunity for sowing the seeds of new ideas in a much wider educational sense: psychological, sociological and philosophical. As has been found, there will be a great temptation to rush to the preparation of new materials, without any prior consideration of aims and possibilities. This is often a sign of a lack of confidence on the part of those involved: tact and expert guidance will be needed if it is to be overcome.

5 THE TRANSFER OF MATERIALS AND "KNOW-HOW"

5.1 Transfer

One of the longest traditions in mathematical education has been that of transferring materials developed in one country, in one particular social context, to another. Thus an account of the geometry known to the Greeks became, two thousand years later, a textbook from which English schools struggled to be free. Two of those on the fringe of the struggle, Hall and Knight, were themselves to become authors of an algebra text on which it could be claimed "the sun never set" – for it was to be used for decades throughout the British Empire. The British saw it as a duty to bring enlightenment and education to others and the use of Hall and Knight, and later Durell, seemed a natural corollary. Unfortunately, such naive views on the value of transfer were to survive the great European empires. Project materials have been translated and transferred with insufficient thought and have later had to be abandoned, for example, the SMSG texts in Taiwan. Now deeper consideration is being given to the problems of transfer and the direct transfer of materials in their original form is rarely attempted at a school level. It does, of course, still take place on a large scale at the university level but by this stage mathematics has become more abstract and its language more international. A book such as Birkhoff and MacLane's *Survey of Modern Algebra* was able, therefore, to exercise great influence on the international development of university mathematics courses in the 1950s and 1960s.

Perhaps, it was the abstract nature of some of the modern school programmes which exercised a certain appeal for some of the countries which hurried to adopt them. To take over new ideas in the teaching of chemistry, physics and biology is financially demanding for it means expensive laboratory equipment. National pride was more easily satisfied by importing an abstract mathematics programme. There was also the argument advanced in the 1960s that by using an abstract, axiomatic approach many of the problems of teacher training would be solved. Given such texts, bright children would not be held back by inadequate teachers. More loosely structured texts would, however, make greater demands on the teacher. The approach proposed was "culture-free". Here, however, we come to the crux of the matter – education, particularly at primary and secondary levels, cannot ever be "culture-free": it is rooted in a particular society. "Culture-free" mathematics would be a largely useless appendix.

If materials are not to be transferred in their original form, then they must be adapted or alternative materials must be written. Both possibilities make heavy demands on manpower which are difficult to meet in many small and developing countries. The idea of a "large-scale" adaptation – one going beyond the substitution of Lagos for London, Dar-es-Salaam for Detroit and pawpaws for pears, is an attractive one and has been used in eastern and southern Africa. The advantages are that the framework is to hand and so the demands on manpower are lessened. The relation is similar to that between "national" and "local" developments proposed in section 3.3, with the essential difference that the gap between the two levels now spans continents and encompasses vast dissimilarities in cultural backgrounds. As a result the "framework" will contain many inbuilt aspirations and assumptions related to, say, pre-school and environmental learning which have no validity in the new context. Such an adaptation, therefore, is not ideal, yet it can represent a step towards a properly designed, indigenous course and provide a means by which future textbook writers begin to learn their trade.

Such transfer difficulties can in certain circumstances arise within a particular country, where distinct cultures have developed side by side, or an alien culture has been imposed alongside an indigenous one. In Canada, for instance, there has been great difficulty in providing assistance to the Inuits and Amerindians. This, however, raises wider issues than those of transfer.

5.2 General assistance

It is one of the sadder aspects of the many attempts to transfer materials that they were made with the best of intentions. They represented a genuine attempt to provide aid. Yet to be of assistance in such a delicate field as education requires a background knowledge of the societies and systems to be helped that very few possess. There is an almost unsurmountable, even subconscious, temptation to impose on others one's own aspirations and patterns of thought. Even when attempts were made to produce original materials specifically for the countries concerned, for example, the Entebbe Project and the Unesco Project for the Arab States, the writing teams were dominated, mathematically and professionally if not numerically, by western authors who lacked any prior understanding of the educational systems of the countries concerned and, more importantly, of the social ethos that was manifested in the schools.

Nowadays there is greater hesitation in suggesting how assistance might best be given. Certainly, it is clear that any curriculum development should be led and controlled by educators from the country in which the development is taking place. Only they can have the requisite background understanding of the problems and be able to communicate with teachers in the desired manner. It may be necessary to nurture and develop such a cadre of potential reformers as a first step. This may mean the provision of training programmes and international visits intended to broaden the mathematical and educational perspectives of the personnel concerned. Regrettably, aid agencies can often sponsor visits only to developed countries whereas interchanges between countries at approximately the same level of educational development are often more valuable.

Until there exists a core of well-informed potential innovators within an educational system any attempts to prompt developments within it are likely to fail from the lack of "middle-management". Expatriates may well still have a part to play in such developments; they have advice and experience to bring, but if their roles become central rather than subsidiary then doubts must be raised about the ability of the reform to sustain itself once they have left the country.

What is gradually being accumulated throughout the world, however, is experience concerning strategies for educational innovation. In particular, for example, IMTEC has for some years attempted to identify patterns of innovation in the OECD countries and is currently hoping to extend its activities to the developing countries. It is likely that a sharing of information and experience concerning strategies of curriculum development and their effects may prove to be one of the most fruitful forms of assistance which one country can provide for another.

6 CONSTRAINTS AND DIFFICULTIES

Throughout this chapter, attention has been drawn to the many constraints and difficulties which face the curriculum developer: problems which arise from pressures from other levels of education, the conservative teacher, the centralized syllabus, and difficulties concerning the allocation of resources, the shortage of personnel and finance.

These are all real problems which must be faced. To ignore them is to run the risk of further disappointments and failures. Yet it can happen that we make too much of the difficulties. For example, that associated with the transfer of children between schools following different curricula seems worse in theory than it often appears to turn out in practice.

Recognition of the difficulties and constraints is, indeed, vital. Yet their existence must not discourage us from seeking to achieve that gradual development of the curriculum that is, and will continue to be, so necessary.

7 TRENDS IN CURRICULUM DEVELOPMENT

Any attempt to delineate trends is bound to have a subjective element – in many cases the "trend" may be more wishful thinking than actuality. Yet, surveying the international scene it would appear that the following general statements can be made with a fair degree of confidence. (It must be realized here that even though one can provide counterexamples to each suggested trend, that, unlike proof where one counterexample is sufficient to disprove a hypothesis, in this instance a single example cannot by itself invalidate a trend.)

It is becoming accepted that:

(a) The provision of materials is not a sufficient condition for the attainment of curriculum development. The provision of new materials is an intermediate step to successful curriculum development – the hard work lies in their successful dissemination and implementation.

(b) In-service education and curriculum development must be more closely integrated. The realization of this has led to greater emphasis on local rather than large-scale projects.

(c) There is a need to design suitable mathematics curricula not only for the academic élite, but for all ability ranges and for the physically and socially handicapped.

(d) Where national educational and social pressures make it necessary, there is a need to produce materials and to develop teaching methods that can be used over a wide ability range (mixed-ability classes).

(e) Although individualization of work via work cards, programmed modules, the computer, etc. has much to offer, it is not by itself the solution to our problems. More attention must be given to investigating the proper role of individualized programmes in the classroom.

(f) Content and method must be considered together (reaction to previous over-emphasis would appear to be generating new biases towards method in secondary schools and towards content in the primary schools).

(g) Increased emphasis must be placed on affective goals and on demonstrating the usefulness of mathematics.

(h) The rethinking of examination procedures and objectives, within a total view of learning and teaching, is an essential ingredient of successful curriculum development.

(i) The role of the teacher is vital. He should become personally involved in curriculum development and should not be subject to unrealistic demands which would only have a disillusioning and dispiriting effect on him. The individual innovator should be cherished.

(j) Pre-service education must prepare teachers for participating in on-going curriculum development. In-service education must aim at fostering the professional growth and autonomy of teachers.

(k) In the design of mathematics curricula, we must not ignore general social and educational objectives and, in particular, the demands of non-mathematicians, such as science teachers, and future employers. Particular attention must be paid to the problems likely to arise at the various "interfaces".

(l) There is a need to consider more closely the processes of innovation and to base curriculum development upon better elaborated theories of teaching and learning and upon improved theories of curriculum development.

(m) Curriculum development must become a gradual cumulative process, rather than a frantic pendulum-swinging exercise.

(n) Curriculum development is the art of the possible, not a yearning for the unattainable.

NOTES, INFORMATION AND REFERENCES

Notes

1. The various sets of initials quoted in the text are explained below in the section "Some abbreviations and addresses".
2. This classification is based on that given by Christine Keitel (Ph.D. Thesis, University of Bielefeld, 1976).
3. See, for example, Chin, R., "Basic strategies and procedures in effecting change", in Morphet, E.L. and Ryan, C.O., *Designing education for the future,* No. 3, New York Citation Press, 1968.

Some abbreviations and addresses

CSMP – Comprehensive School Mathematics Program, 610 East College St., Carbondale, Illinois 62901.

CUPM – Committee of the Undergraduate Program in Mathematics, P.O. Box 1024, Berkeley, California 94701.

IMTEC – International Management Training for Educational Change, Postboks 79, Blindern, Oslo 3, Norway.

IMU – Individualiserad Matematikundervisning - Individualized Mathematics Teaching; information obtainable from School of Education, Fack, S-20045, Malmö 23, Sweden.

IOWO – Instituut voor de Ontwikkeling van het Wiskunde Onderwijs – Institute for the Development of Mathematics Instruction, Tiberdreef 4, Utrecht/Overvecht, The Netherlands.

IPI Mathematics – Individually Prescribed Mathematics Instruction.

IREMs – Instituts de Recherche sur l'Enseignement des Mathématiques – Institutes of Research on Mathematics Teaching

Madison Project, 1210 West Springfield Ave., Urbana, Illinois 61801.

MMCP – Mathematics for the Majority Continuation Project (Schools Council).

NSF – National Science Foundation, 1800 G.St., NW Washington DC 20550.

RECSAM – Regional Centre for Education in Science and Mathematics, Glugor, Penang, Malaysia.

SEED – Special Elementary Education for the Disadvantaged, 2336-A McKinley Avenue, Berkeley, California 94703.

SMP – School Mathematics Project, Westfield College, Kidderpore Ave., London, NW3 7ST.

SMSG – School Mathematics Study Group, USA; project materials published by L.W. Singer, Random House and Stanford.

Schools Council, 160 Gt. Portland St., London, W1N 6LL.

TEMLAB – Papua, New Guinea.

USMES – Unified Science and Mathematics for Elementary Schools, Educational Development Center (EDC), 55 Chapel St., Newton, Massachusetts 02160.

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PANEL

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Chapter VIII

Methods and results of evaluation with respect to mathematics education

Jeremy Kilpatrick

> *The convenience of the examiner, and even precision in the results of examinations, ought unhesitatingly to be sacrificed when they are in conflict . . . with the vastly more important interests of education. Of the many evils which our examination system has inflicted upon us, the central one has consisted in forcing our school and university teaching into moulds determined not by the true interests of education, but by the mechanical exigencies of the examination syllabus.*
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> *E.W. Hobson, 1910 (1)*

INTRODUCTION

Questions of evaluation abound in mathematics education. How well can Johnny add? Is the new mathematics syllabus better than the old one? Is Miss Jones a better mathematics teacher now than she was five years ago? How well are the pupils in our school learning mathematics? How well prepared are they to use mathematics in the future? Questions such as these can be made more precise, but they cannot be avoided. People are involved in the educational process, and people want to know how the process has proceeded and what its results have been.

1 THE PROCESS OF EVALUATION IN EDUCATION

The process of evaluation is in essence quite simple and straightforward. Evaluating is making a judgement as to the value or worth of something. Given an object to be evaluated, one chooses a dimension of value, gathers information about the object relative to the dimension, and applies a scale of value to the information obtained. In theory, the process is simple. But in educational practice, it is never simple – for two basic reasons.

In the first place, the "object" to be evaluated is neither homogeneous nor static. A pupil's ability, a teacher's competence, a curriculum's adequacy – these things are complex phenomena, changeable over time and across situations. Even when the objects to be evaluated are physical objects such as textbooks and instructional aids, one is typically interested in an extrinsic as well as an intrinsic evaluation: the question is not simply how good is this object, but how well does it work. A textbook is not like a refrigerator or a car. It cannot be tested under laboratory conditions and then warranted to perform as specified in every school or classroom. Its effectiveness in promoting learning is highly situational, greatly influenced by the teacher who uses it, the pupils who study from it, and the instructional setting in which it is used. The heterogeneous, dynamic nature of educational objects creates problems in evaluation: one must somehow sample occasions and situations, and one must find or create instruments to assess the object along various dimensions of interest. These assessments can then be weighted to yield an evaluation.

But what are the dimensions of interest? And to whom are they of interest? These questions go to the heart of the second reason why educational evaluation is so complicated: educational evaluation, like education itself, is a socio-political process. Evaluations are undertaken because someone must make a decision.

Johnny wants to go to the university; does he know enough mathematics to qualify for entrance? The university authorities, Johnny's teachers and parents, and Johnny himself may know very well that Johnny's knowledge and understanding of mathematics is not a seamless web: he knows some concepts, but not others; he can solve some kinds of problems, but not others; he understands some principles, but not others. Johnny has a profile of abilities, and if one were to measure various aspects of his mathematics achievement at any time, one would obtain a vector of scores, not a single score. Such a vector, together with information on how the components were changing, might be extremely useful to Johnny and his teacher in deciding what mathematical activities Johnny might engage in next. But society presses for a single score because binary decisions must be made: Does Johnny enter University X or not? Does Johnny qualify for a diploma in mathematics? Does Johnny earn a passing mark in this mathematics course or on that examination?

A supervisor may know very well that Miss Smith's teaching ability is a complex composition of many and variable parts. During her supervised teaching practice, Miss Smith may have been consistently rated by observers as good in dealing with pupils' misbehavior, but tests, questionnaires, and observers' ratings may have suggested that she was consistently poor at explaining new concepts to pupils and sometimes good and sometimes poor at engendering in her pupils a liking for mathematics. The supervisor can evaluate Miss Smith's teaching ability on each of several dimensions. But somewhere, sometime, decisions must be made, based to some extent on the supervisor's evaluation: Should Miss Smith be certified to teach mathematics? Should she be chosen for a teaching position?

A textbook selection committee must recommend a set of mathematics textbooks from which teachers may choose. They are aware that Textbook A, produced at considerable expense by a special curriculum development project and extensively field tested in a small number of schools, was very successful in introducing pupils to techniques of solving mathematics problems. Although the project's evaluation studies showed that the pupils in the field-test schools had satisfactory achievement as measured by final examinations and standardized mathematics achievement tests, committee members have heard reports that teachers in the field-test schools thought Textbook A gave insufficient attention to drill and practice. Further, the committee is worried that many teachers and pupils would have trouble understanding the materials without the special help given by the project in the field-test schools. The committee is aware that Textbook A has both strengths and weaknesses and that, depending on how the teacher uses it, children will learn different things from it and develop different attitudes toward mathematics. Should Textbook A be added to the list? What information is needed to make this decision, and how will the information be used?

When one stands close to the educational object to be evaluated, one sees its complex and dynamic qualities. When one steps back – as one must to make a decision – these qualities are lost and a single scale of value comes into play. The conflict between these two points of view – especially when they are taken by different sets of people – accounts for many of the problems in, and most of the dissatisfaction with, current evaluation practice.

Consider, for example, the issue of external examinations at the end of secondary school. The last decade has seen a trend in most countries away from schemes in which external examiners both set and grade a terminal examination toward schemes in which teachers set and grade their own examinations and make informal evaluations more or less continuously throughout the year. This change has been, for the most part, welcomed by students, who feel that the newer schemes yield a "fairer" assessment of their learning. Students are sensitive to questions of evaluation, and they resent being judged, especially by people who do not know them and their work.

Recent student protests in many universities have dealt not with broader social issues but with issues concerning university policies – and evaluation has been a popular theme. For example, Anderson (1975, p.187) reported that recently in Australia, "in at least three universities where students occupied the administrative buildings, the central issue was assessment. Feedback of the results of their learning is acceptable to students, but the reduction of their work to a cypher, ranked in order of performance, is a violation of quite deeply held values".

But as external examinations are replaced by internal assessments, questions arise as to how to compare the learning of students who have had different teachers and studied in different schools. Why, one might ask, need their learning be compared? The answer is that as long as colleges and universities do not have places for all who apply, some "fair" method must be used to decide who will be admitted and who will not. If teachers are unable or unwilling to assign a single mark to a student's work, then either some way must be found to reduce the teacher's evaluation to a "cypher" or the evaluation will be disregarded. Even if teachers do assign a single mark, one needs to worry about the comparability of different teachers' ratings. Researchers in Toulouse (Cransac and Dauvisis, 1975) have shown how teachers can vary even in grading the same external examination. The problem is compounded when the examinations or assessments are not the same. In the Federal Republic of Germany, differences between *Länder* in average scores on the (internally set and graded) *Abitur* are eliminated by a norming process. But the process cannot adjust for differences between teachers in their marking practices, and consequently, pressure has grown for a national set of rules for teachers to follow in setting and grading examinations (Lunschberger et al., 1975). In the Netherlands, it has been proposed that all information about the student's performance in secondary school and on examinations be disregarded in favour of a lottery for entrance to university programmes. Although existing schemes for assessment and examination are hardly perfect, they certainly can predict better than chance a student's success in the university.

There is an inevitable tension between the "firing line" and the "headquarters" conceptions of evaluation. Technically, evaluation problems reduce to obtaining representative samples of information and then resolving the information into a weighted score, minimizing the amount of information lost and maximizing the comparability of information gathered from different sources. A great deal of energy has been spent, and some knowledge accumulated, over the past two decades in attacking these technical problems. But deciding what information to obtain, what scales of value to use, and how to weight various scales to arrive at a decision – these are not technical problems. And they are problems that are approached differently by different participants in the evaluation enterprise.

The thesis of this chapter is that educational evaluation is an interactive process, involving the knowledge, values, and beliefs of those whose accomplishments, past or potential, are being evaluated as well as the knowledge, values, and beliefs of those who do the evaluating and those on whose behalf it is done. It is a psychological process because it affects people's beliefs about themselves, what they are competent to do, and the importance of what they do. It is a socio-political process because it leads to decisions concerning who shall be educated, who shall teach them, and what shall be taught. In democratic societies, evaluation should itself be democratic, yet recent actions by ministries of education and other educational authorities suggest a strengthening of what has been called the bureaucratic approach to evaluation (MacDonald, 1976). Examples could be cited from several countries where bureaucrats have used evaluation activities to legitimate decisions already made on other grounds.

As one surveys trends and issues in evaluation with respect to mathematics education, one should strive constantly to clarify the purposes served by an evaluation and the decisions to be made and actions to be taken on its basis. One should give special attention to the balance between the individual's right to develop uniquely as a student or teacher of mathematics and the society's need for citizens with mathematical competence. A first step in achieving this balance is to make clear the various conceptions of evaluation held by the participants in the process. An examination of some current issues in evaluation with respect to mathematics education may help one see some of the lenses through which people have been looking at evaluation problems.

2 SOME CURRENT ISSUES IN EVALUATION WITH RESPECT TO MATHEMATICS EDUCATION

2.1 The "failure" of the "new math"

The last five years have seen an increasing number of articles and books asking whether, or asserting that, the new mathematics curriculum programmes of the 1950s and 1960s have "failed". A new generation of curriculum developers, textbook writers, and critics has emerged, many of whom stand not so much on the shoulders of the preceding generation of reformers as on their necks.

In the United States, the Conference Board of the Mathematical Sciences appointed a National Advisory Committee on Mathematical Education (1975), whose report is largely a reply to the claims of Morris Kline and others that the "new math" was a disaster. In the United Kingdom, the School Mathematics Project has come under attack for its alleged lack of attention to applications of mathematics. In the Federal Republic of Germany, the "new math" has become a political issue in some *Länder*, with the party out of power promising to junk the new syllabus once in power. Several other countries have witnessed – if not attacks by politicians – complaints by parents, by prominent mathematicians and scientists, or by teachers' organizations concerning new syllabuses and new textbooks.

A great many evaluation studies have been undertaken to try to get evidence on how effective the new curriculum programmes have been. It is not possible to analyse the results of these studies in any detail; one can say only that the vast majority have been both inconclusive and ineffective.

The typical study has been concerned with what pupils have learned as a result of the new curriculum programme, and the typical design has been to compare the new and the old programmes in some approximation to an experiment on the effectiveness of a drug in medicine or a fertilizer in agriculture.

Putting aside the question of how to achieve in schools the random assignment of "subjects" to "treatments" that one would want in such an experiment, one can see that if a programme relies on the teacher for implementation, one must have a large enough sample of teachers so that differences in how the teachers implement the programme do not obliterate differences between the programmes. Few evaluation studies have involved more than a handful of teachers and classes. And, of course, the larger the sample of teachers one has, the harder it is to determine whether each programme is being implemented according to its specifications.

A survey of the literature on mathematics curriculum evaluation suggests the following wry advice for the evaluator who wishes to undertake a "typical study." First, keep in mind two key axioms of mathematics learning:

(a) On the average, pupils learn the mathematics they have been taught.

(b) On the average, pupils do not learn the mathematics they have not been taught.

If you want to demonstrate that the new programme is superior, give a test on mathematics contained in the new programme but not in the old. If you want to demonstrate that the old programme is superior, give a test on the mathematics contained in the old programme but either not present or not emphasized in the new. If you want to demonstrate that there is no difference between the two programmes, give a test on mathematics contained in neither or both.

Second, keep the programme short; use small, non-random samples of teachers, pupils, and classes; and give either unreliable tests or tests of uncertain validity. Then by means of powerful statistical techniques, you can obtain results capable of interpretation any way you choose.

This caricature is, unfortunately, not too wide of the mark. Most evaluation studies have been simply inadequate for answering the questions they were set up to answer. One is led to ask whether these studies may have been based on false or inadequate conceptions of the evaluation process.

2.2 Changes in examination systems and testing practice

The last two decades have seen the extension of comprehensive mass education to the upper stage of secondary school in many countries of the world. This extension has sometimes continued into higher education: for example, both Italy and New York City have attempted essentially to guarantee the secondary school graduate a place in the university (2). The expansion of secondary and higher education has occurred differently in different countries, but every country has been faced with providing a greater variety of educational programmes than was the case before. The role of evaluation has tended to shift somewhat from the *selection* of pupils for further education to the *allocation* of pupils to different instructional programmes.

As long as one needs only to skim the cream off the crop of pupils, almost any difficult examination will suffice. A large number of capable pupils are likely to fail the examination, but few incapable pupils are likely to pass it. When one is obliged to educate a relatively large fraction of the population, however, then one ought to be more concerned with *diagnosing* a pupils' mathematicial abilities and achievement than with *comparing* one pupil's abilities or achievement with another's. Unfortunately, the traditional use of mathematics as a selection device appears stronger now in some countries, such as France and Israel, than ever before.

Examination systems and testing practice are being modified, however, to take account of new mathematics programmes and increased differentiation of instruction (Peterson, 1975). In England and Wales, alternative modes of the Certificate of Secondary Education (CSE) examination allow a school to set up its own syllabuses, courses, and (externally moderated) examinations. The purpose of the CSE is to describe a pupil's achievement rather than simply to indicate "pass" or "fail", and therefore course work and teacher judgement are used in an attempt to assess over a longer period and under more natural conditions (Mathematical Association, 1974). The spreading of assessment over a longer period is also seen in the reformed *Abitur* of the Federal Republic of Germany.

An almost universal trend in constructing examinations and tests has been to spell out, at some level of specificity, the objectives whose attainment is to be measured, and then to write questions or test items designed to assess the attainment of these objectives. The test constructor typically uses a "matrix" model of classifying objectives, and hence test items, according to two dimensions. One dimension is the mathematical content with which the objective or item is concerned, and the other dimension is the cognitive process (knowledge, comprehension, application, etc.) specified in the objective and presumed to be used by the pupil in answering the item (Wilson, 1971). Accompanying the increased use of a matrix model for designing examinations and tests has been a trend for questions and test items to become less open-ended and more objective, in the sense that two people independently grading a pupil's answer would give it the same score.

The attempt to be more explicit as to the mathematical content and cognitive processes involved in an examination question or test item has, on balance, proved quite useful. It can be a very illuminating exercise for a teacher first to write down the objectives with which a test will be concerned and then to check that the test items he or she writes are in harmony with the objectives. It is also revealing to classify the objectives of existing tests and examinations, thereby indicating the test-makers' assumptions as to what is important. The classification of test items has permitted the construction of "item banks": pools of test items classified according to various schemes from which a teacher or other evaluator may draw sets of items to form tests with specified characteristics (3). When items have been classified according to a content-by-process matrix, instead of receiving a single score on a test, a pupil can be given a profile of scores, one for each of several categories.

In some quarters of mathematics education, however, a reaction has set in against the use of such classification schemes, and especially the schemes for classifying "processes" or "abilities". Arguments are raised that such approaches lead to overly simple, if not trivial, test items instead of complex questions that might test a variety of skills and abilities.

The most commonly used classification schemes are based on Bloom's taxonomy of educational objectives, which was developed out of the experience of American college and university examiners in devising objective test questions to measure different types of instructional objectives. It was not designed with any one subject field in mind. As Ormell (1974, p.7) has noted in a penetrating critique of the taxonomy, Bloom's categories of behaviour "are extremely amorphous in relation to mathematics. They cut across the natural grain of the subject, and to try to implement them – at least at the level of the upper school – is a continuous exercise in arbitrary choice". Numerous alternative taxonomies have been proposed for mathematics, but none has found wide acceptance. Criticisms by mathematicians have sometimes been forestalled by including a high-level category with "creative", "intuitive", or "insightful" in the title, but such measures are little more than cosmetic.

The deeper criticisms of Bloom's taxonomy (and others based on it) go beyond the question of whether a given hierarchy of processes is complete or ordered correctly to questions such as: (a) Should the cognitive "domain" of behaviour be distinguished from the affective and psychomotor "domains"? (b) Is it reasonable to postulate a linear order for processes? (c) Is process independent of content? Perhaps the most penetrating question one can ask is whether isolated objectives (and isolated test items) can be classified once and for all or whether they have meaning only in relation to a particular curriculum or instructional programme (Freudenthal, 1975, p.163).

The matrix model has dominated the thinking of evaluators for two decades. It is a useful lens through which to view certain problems of test construction. The challenge now is to find other lenses that might be better suited to contemporary problems of evaluation in mathematics education.

3 METAPHORS FOR EVALUATION

3.1 The dominant metaphor

The idea that education is a production process, with inputs and outputs, the idea that a curriculum or an instructional programme can be evaluated against a list of specifications; the idea that behaviour can be broken down into "behaviours" (with knowledge broken into "knowledges" and understanding into "understandings"); the idea that teaching is no more than the shaping of these "behaviours", that learning is no more than a change in them, and that evaluation is no more than measuring which ones have changed and by how much – these are expressions of the dominant metaphor today for addressing questions of educational evaluation: the metaphor of the evaluator as consulting engineer.

This way of thinking about evaluation began in the United States, where the conception of education as production fits the ethos of the society, but like Coca-Cola, the engineering metaphor now can be found world-wide. Although the metaphor has many sources, one can see it clearly in Ralph Tyler's concept, developed in the 1930s, of education as the facilitation of changes in pupils toward desired goals and in Bloom's taxonomy, developed in the 1950s, which stressed the careful delineation of goals and the careful measurement of goal attainment. The approach is sometimes characterized as "behaviouristic", but this characterization misses the point. As Ormell (1974) has observed, the problem is not the insistence that pupils' achievement be publicly verified in terms of behaviour; it is the tacit assumption that easily measured things are more "real" than things that are more difficult to measure. This assumption is not essential to the engineering metaphor (and Tyler, for one, did not make it). It is perhaps natural if one conceives of one's role as engineering, however, to pay most attention to those phenomena for which one has measuring instruments at hand and to pursue specified outcomes while neglecting unanticipated side effects (Atkin, 1973).

The tendency to conceive of education as a product capable of being decomposed into small, measurable components can be seen in the United States in some programmes of "individualized instruction" and in collections of "instructional objectives". It can also be seen in some teacher education programmes that are "performance-based" or "competency-based".

3.2 Other metaphors

Dissatisfaction with the engineering metaphor, particularly as it applies to curriculum evaluation, has led to alternative ways of thinking about the evaluation process. Evaluation has been conceived of as medical diagnosis, as literary criticism, as art criticism, as anthropological field study, as criminal investigation, and as legal argument. The role of the evaluator has been described as that of clinician, critic, participant observer, passive observer, detective, juror, and attorney, among others (see Anderson et al., 1975). The details of these metaphors are less important than their intent, which is to overcome the limitations of the engineering metaphor by providing alternative ways of conceptualizing and then approaching the problem of evaluation. The alternative metaphors have arisen because of concern about such questions as: Should evaluation seek causes as well as effects? What should the evaluator's relation be to the people whose work is being evaluated? How can one obtain a richer description beyond that provided by numerical scores? How can one describe both process and product (assuming the distinction is meaningful)?

The argument advanced in this chapter is not that any of these alternatives is better than the engineering metaphor; rather it is that only by becoming aware of alternative conceptions can one see the limitations of any single conception. Circumstances alter cases, and the circumstances of educational evaluation today demand a variety of approaches.

4 THE ILLUSION OF CURRICULUM EVALUATION

Most definitions of "curriculum" stress that it is more than a syllabus or an outline of the subject to be taught. The term is used more broadly to characterize the organized learning experiences that a school provides for pupils, and it is seen as involving issues of objectives, content, instructional methods, and assessment methods. Under this broader conception one cannot say, for example, that a country has a uniform curriculum in mathematics for pupils in grade five. Although there may be a national syllabus, the implementation of this syllabus will depend upon such factors as the teacher, the school, the pupils, and the instructional methods and materials. Even to identify and describe the mathematics curriculum for a given grade in a given school may turn out to be an exceedingly difficult task.

Consequently, it is an illusion to think that one can evaluate a curriculum in any global sense. The curriculum does not exist globally; it exists only in the specifics of a particular instructional setting. Failure to appreciate what might be called the "situation-specific" nature of the curriculum may account for much of the current confusion about questions of evaluation. Attempts to ignore situational variation in curriculums usually lead to the search for a "least common denominator" to be evaluated, which can have constrictive effects on subsequent instruction.

It does make sense, however, to talk about evaluating the activities and products of a given curriculum development project. This may seem like a small difference – between evaluating a curriculum and evaluating a project's work – but it is an important distinction to make. When one attempts to evaluate a "curriculum" *per se*, one tends to reify it and to lose sight of its situation-specific character. One begins to talk of its effectiveness – as though it had such a quality – and to set up studies to compare the effectiveness of various curriculums. Such studies inevitably encounter difficulties because they assume that curriculum effectiveness is a quality that can be measured by, say, a set of tests and examinations.

If one sets out to evaluate the activities and products of a curriculum development project, on the other hand, one has a different perspective. The activities of a project concern how it is planned, initiated, and organized; its working methods; how materials are produced and disseminated; and its internal evaluation methods. How are such activities to be described and evaluated? Obviously, they are not susceptible to investigation by controlled experimental designs from the psychological laboratory; they demand the skills of the reporter, the historian, the anthropologist, and the critic. The products of a curriculum development project include not only whatever instructional materials it may produce but also guides for teachers; reports to parents, administrators, and funding agencies; scholarly books and articles; and a variety of working papers and internal documents. Such non-instructional material ought to be considered in any comprehensive project evaluation, and the evaluation of this material too requires special skills (Eraut, 1976).

The instructional products of a curriculum development project may include such things as textbooks, films, activity packages, and instructional modules. Instructional material can be evaluated by applying criteria related to mathematical correctness, ease of use by pupils and teachers, clarity of exposition, and so on. When questions of "effectiveness" arise, they can be seen as concerning only a part of the evaluation problem.

What does it mean for a mathematics textbook to be "effective"? Effective in teaching what? To whom? Under what circumstances and as part of what instructional programme? Questions of effectiveness are always relative, always situation-specific.

Experience in many countries over the past two decades in attempting to evaluate the work of curriculum projects in school mathematics has shown again and again the impossibility of obtaining definite answers by empirical means to questions of effectiveness, even when such questions are made more precise. Is Textbook A more effective than Textbook B in teaching third-grade pupils to multiply whole numbers? The answer, of course, depends on which set of pupils is meant. Unless the set of pupils is clearly specified, the question cannot be answered. But if the set of pupils is specified, the answer is uninteresting in itself. One wants to be able to generalize the answer to other sets of pupils in other places at other times. The answer also depends on what is taken as evidence of effectiveness – and on this rock many evaluation studies have foundered. To avoid ambiguity, the criterion of effectiveness is usually chosen to be quite narrow: for example, knowing how to multiply whole numbers may be operationally defined as being able to do 95 percent of the multiplication problems in a given set. On the one hand, ambiguity is never entirely eliminated. Are the problems given orally or in written form? Are the answers to be in the same form? How much time is allowed? Are the pupils told to try as hard as they can? Are they told to guess at an answer if they don't know? When is the test given: immediately after instruction or some months or weeks later? The questions go on and on – some of them important, some not, but each indicating that ambiguity is still present. On the other hand, too narrow a criterion of effectiveness can be misleading. Textbook A may be overwhelmingly more effective than Textbook B with a given set of pupils when the criterion is as indicated. Suppose the pupils who use Texbook A, however, have little understanding of the operation of multiplication, become hopelessly confused when introduced to division and consequently lose their skill in multiplication, are unable to solve problems involving multiplication, and are led to turn away from mathematics in digust. Then even though in a narrow sense Textbook A has been relatively effective in teaching multiplication of whole numbers, in a broader sense it has been ineffective.

Evaluators or mathematics curriculum projects are increasingly coming to realize the inadequacy and imprecision of the "effectiveness" concept. They are beginning to understand that once an evaluator and the "consumers" of the evaluation have agreed on both the purpose of the evaluation and the object to be evaluated, the evaluator's real task is to gather evidence from a variety of sources so as to describe the object from various points of view. If the object's "effectiveness" is at issue, then the evaluator cannot be expected to resolve the issue. At best, he or she can supply evidence concerning plausible alternative explanations as to how the object functions under specified conditions.

The movement toward a broader conception of the evaluation of a curriculum project's work has occurred more in theoretical discussions of evaluation than in actual evaluation studies. Some evaluators of mathematics curriculum projects, however, have gone beyond the "testing" mode of evaluation to such techniques as classroom observations, analyses of materials, job analyses, parent questionnaires, case studies of individual pupils, site visits, and interviews with teachers, project personnel, and administrators. These evaluators include, for example, Herbert (1974) and Larsson (1973), who made imaginative and resourceful use of information from various sources, including interviews, observations, and questionnaires. The case study methodology has been used effectively by Erlwanger (1975) to examine the influence of the mathematics curriculum on children's conceptions of mathematics.

Of course, it is still possible to make the case for curriculum reform on the basis of higher scores on achievement tests used previously (Kondakov, 1974). The trend, however, is to borrow or create a variety of new tests and inventories. Much attention has been given in recent evaluation studies to assessing "non-cognitive" aspects of mathematics learning, including anxiety, self-concept, impulsivity, perseveration, fatiguability, and achievement motivation (Klein, 1974; Radatz, 1976; Weis, 1973; Weis and Bauersfeld, 1973).

5 PROGRAMMES OF ASSESSMENT IN MATHEMATICS

As national, state, and local education authorities have increased their investment in educational programmes in general, and curriculum reform projects in particular, increased attention has been given to the question of accounting for these expenditures. The notion of "Gross Educational Product", analogous to a country's Gross National Product, has been proposed, and achievement in mathematics has been seen as a part of the Gross Education Product. In the United States, the National Assessment of Educational Progress has made several surveys of children's and adult's achievement in school subjects. Achievement in mathematics was surveyed in 1972–73 and will be surveyed again in 1977–78. In the United Kingdom, the National Foundation for Educational Research began a project on Tests of Attainment in Mathematics in Schools (TAMS) in 1972 to develop banks of mathematical questions that would be used in nationwide surveys of the attainments of 11- and 15-year-old pupils. A special effort has been made to devise new types of questions to test understanding of mathematics in the broadest sense, including attitudes and abilities.

The first International Study of Achievement in Mathematics in 1964, although a research study in comparative education and not an attempt by authorities to account for educational expenditures, also took a Gross Educational Product view of mathematics achievement. A stinging critique of this study has been delivered by Freudenthal (1975), who correctly notes the failure of the organizers of the first International Study to deal adequately with variations between countries in the mathematics curriculum (4). A second International Study is being planned, and one hopes it will include variables relating to curriculum and instruction so that the results will be of greater interest to mathematics educators.

The Gross Educational Product view of mathematics achievement is but another manifestation of the engineering metaphor: schools are factories, entering pupils are the raw material, graduates are the finished product, and what they know as measured by tests and examinations is the value of the product. The dangers in this view reside not only in the faith placed on tests but also in the interactive nature of testing. It is in the nature of tests that although they may have been designed for one purpose, they come to be used for other purposes.

When, in calculating the Gross National Product, economists total up the value of a given factory's output, the measurement process itself has no effect on next year's output. But when an educational authority sets up a battery of tests or an item bank and administers tests to groups of pupils on a regular basis, the test items come to be used by teachers and pupils as indicators of what should be learned in school. In fact, the educational authority ordinarily intends that this should happen. The economist and the factory owner agree that monetary units define the scale of value; the economist wants to measure, the factory owner to maximise, output as measured in these units. But the educational authority and the pupils and teachers may not agree on the scale or scales of value – who is to prevail?

6 THE EVALUATION OF LEARNING IN MATHEMATICS

Examinations and tests predominate in all countries as instruments for measuring how much mathematics pupils have learned, and this predominance is likely to continue indefinitely. The nature of examinations and tests is changing, however, in line with trends noted above.

The use of oral examinations to assess mathematics learning seems to be declining somewhat. In the Soviet Union, oral examinations in geometry are still given in grades 8 and 10, but some thought has been given to changes in the system to reflect the new syllabus (Owen and Watson, 1975). In the United Kingdom, where they have been traditionally used with university students whose examination marks are equivocal, oral examinations are used less commonly now with undergraduates, although their use with graduate students continues. In the Federal Republic of Germany, oral examinations play a minor role in the reformed *Abitur*, but they are still the dominant form of university examination in most European countries. A problem with oral examinations is that unless all examinees are asked the same questions, it is difficult to compare one examinee's performance with that of another; whereas if a single set of questions is used, one loses the flexibility of the oral examination and its power to diagnose an examinee's strengths and weaknesses.

As systems of external examinations shift to some mixture of external and internal examinations, the examination results tend to be used for diagnostic and formative purposes (indicating how further instruction might proceed) in addition to the traditional normative purpose (indicating how the examinee's performance compares with that of other examinees). Diagnostic and formative evaluation, however, suggests assessment that is both more frequent and more detailed than traditional examination schemes have permitted. Almost all countries have seen an increase in the extent and frequency of objective testing in mathematics, with brief tests on rather specific topics being used both to monitor progress and to guide instruction (5).

Traditional psychometric methods of test construction were developed to serve normative purposes. Consequently, the use of these methods has led to tests composed of items selected for their ability to discriminate pupils who have learned from pupils who have not learned. More recently, as tests have been used increasingly to judge whether a pupil has achieved "mastery" – without reference to how other pupils are doing – new methods of test construction have been developed. So-called "criterion referenced" tests (Frey, 1975; Klauer, 1974; Millman, 1974) are being used to assess whether a pupil has achieved a pre-determined level of performance, usually with reference to a specific educational objective.

Criterion-referenced measurement has been criticized not only because of the tendency to test minutiae but also because of the need to set pre-determined levels of performance. Again, the engineering metaphor is at work: The car should be capable of consuming only 8 liters in 100 kilometers; the secretary should be capable of typing 60 words per minute; the pupil should be able to solve correctly 8 out of 10 problems. Why 8 problems? Why not 3? Or 10? It is as much of an illusion to think that 80 per cent correct is not a normative criterion as it is to think that a car consuming 8 liters in 100 kilometers would still be considered economical if all other cars were using 4 liters or to think that 60 words per minute would still be a minimum requirement in a land where secretaries were one-handed. Furthermore, criteria such as 80 per cent correct are predicated on the assumption that all problems are equivalent; it does not matter which ones are solved correctly. Not all mathematics educators view mathematics learning as capable of decomposition into components measured by sets of equivalent questions or problems.

In many respects, psychometric theory has become elaborated well beyond the capability of practitioners to use it. It demands, paradoxically, considerable mathematical sophistication to understand it and a considerably narrow view of mathematics and its learning to be willing to apply it.

Although the use of tests seems to be growing in most countries, attempts to make relatively continuous assessments of pupils' learning have sometimes led to reactions against "over-testing". Such reactions, coupled with dissatisfaction with existing tests and examinations, have led to experimentation with alternative forms of assessment. Interviews with pupils according to Piaget's methods have been used in Geneva to assess pupils' learning under experimental curricula, and the Nuffield Mathematics Project devised "check-ups" along the same lines. In the United States, Project TORQUE (Tests of Reasonable Quantitative Understanding of the Environment) is attempting to devise tests that can be administered like games and that will be more useful to teachers than standardized achievement tests have been (Education Development Center, 1976). In the United Kingdom, the Sixth-Form Mathematics Curriculum Project has prepared an alternative Ordinary level examination for its Applicable Mathematics course that permits candidates who are stumped by a question to tear off a paper revealing a hint that may help them. The questions are arranged in a sequence, and candidates can check to see whether their answers so far are correct, thus making the examination more of a learning process.

A common complaint in the literature on evaluation in mathematics education has been that we lack instruments to measure such constructs as problem-solving ability and creativity in mathematics. We know very well how to measure a child's ability to do numerical computations, according to the argument, but we know very little about measuring the child's insight into mathematics, mental attitude toward mathematics, ability to respond creatively to mathematical ideas, or ability to formulate and solve challenging mathematical problems. Certainly if more instruments were available to measure these constructs, they could be used by teachers to measure a child's mathematical development as well as by evaluators of curriculum projects to measure how successful the project had been in developing these qualities in children.

Progress has been slow in developing such measures, but the past five years have seen some promising work begun. In Japan, there has been interest in constructing measures of creative attitude and open-ended problems with multiple solutions (Fukabori, 1973; Shimada, 1974). In the United Kingdom, the TAMS project mentioned in section 5 is developing tests on practical mathematics, "creative mathematics", style of understanding, visualization, and attitude to mathematics. It is notable that both the TORQUE and the TAMS projects have done intensive pilot-testing work with small groups of children and with a variety of formats for questions in order to discover how the children are thinking about the questions. This concern for validity has not always been a prominent feature of test construction, especially with respect to attitudes. Gardner (1975) has made a valuable critique of flaws in recent attitude assessment instruments, and Hoyles (1975) has suggested an interview technique for assessing attitude and motivational factors.

7 THE EVALUATION OF MATHEMATICS TEACHING

There are few trends to be seen in the evaluation of mathematics teaching. The dominant mode of direct evaluation continues to be – as it has been for decades – for some supervisor or inspector to observe a sample of the candidate's teaching and then to give it a rating based on "professional judgment", that is, an amalgamation of other teaching he or she has seen and a private vision as to what good teaching ought to be. In addition, teaching is evaluated indirectly: one's credentials, degrees, diplomas, recommendations, and performance in an interview are taken as indicators of teaching competence.

Observations of teaching today tend to be somewhat more sophisticated than a decade ago because various techniques for simulating classrooms, making observations, and recording behaviour have been developed that permit more detailed analyses of teaching. Research has not convincingly shown that any pattern of teaching behaviour leads consistently to better learning, however, so although teaching behaviour can be analysed more thoroughly, the problem of distinguishing "good teaching" from "bad teaching" seems to defy resolution.

There have been some attempts in the United States to link the licensing of teachers (and sometimes salaries and job security) to evidence of "teaching effectiveness", as measured by pupils' achievement test scores and evaluations by supervisors and pupils. This effort is part of the larger movement toward performance-based or competency-based teacher education already noted (6). Most countries, however, show no signs at present of joining this movement.

8 THE EVALUATION OF MATERIALS FOR MATHEMATICS TEACHING

Most countries have in recent years witnessed an increase in the variety of materials that are available for mathematics teaching. Of course, when official syllabuses are spelled out in great detail, as many are coming to be, only a small number of textbooks (perhaps only one, if the officials and the textbook authors collaborate) may fit the syllabus. Teachers seldom have complete freedom in fixing syllabuses and selecting textbooks. Depending on the budget, however, they may be able to purchase additional non-text-book materials. Furthermore, some materials – such as pocket calculators – may be purchased by the pupils themselves.

When teachers are free to choose instructional materials to be purchased, they are usually bewildered by the choices available to them. How are they to know which materials are effective? For reasons developed above in sections 3 and 4, the notion of "effective materials" – like the notions of "effective teacher" and "effective curriculum" – is a fantasy. It stems from the engineering metaphor and a blindness to the situational variability of instruction.

Mathematics educators are beginning to realize that what is needed are fuller descriptions of what the materials are, with some indication of how they might be used. In France, the Strasbourg and Toulouse IREMs have developed textbook analysis grids. In the United States, the Educational Products Information Exchange (EPIE) has analysed both mathematics textbooks and materials for individualized instruction in mathematics. Reviews of materials in journals and publisher's advertisements appear to be the main sources of information on new materials. One problem with reviews is that the reviewer may be out of touch with the audience of teachers. A study in the Federal Republic of Germany showed that professors of mathematics education and elementary school teachers ranked elementary mathematics textbooks in very nearly the opposite order.

9 SUMMARY AND CONCLUSION

The chapter opened with the argument that educational evaluation is a complicated process because: (a) the objects of evaluation are dynamic and heterogeneous, and (b) the process of evaluation is interactive, psychological, and, perhaps most important today, socio-political. This theme was developed through a discussion of two current issues in evaluation: the "failure" of the "new math" and certain changes in examination systems and testing practice. The discussion suggested that at least some of the dissatisfaction with current evaluation practice arises from a too-narrow view of what evaluation should be and what kind of information it should provide. The limitations of an excessive commitment to an engineering metaphor of education, and hence of educational evaluation, seemed to be the source of many problems, and alternative metaphors were cited. Then attention was given to some of the objects that one might attempt to evaluate. The mathematics curriculum, as an object, was portrayed as highly dependent on specific situations. The idea of evaluating a "curriculum" was rejected in favour of the idea of evaluating the activities and products of a curriculum development project. Local, state, national, and international programmes of assessment in mathematics were seen as, for the most part, embodiments of the engineering view, and a broader view was urged. Other objects of evaluation are the pupil's learning in mathematics, the teaching of mathematics, and the materials that are used for mathematics instruction. Each of these objects presents its own problems to the evaluator, but the need for a broad perspective was shown in each case.

The only way a discipline like mathematics education can develop is to find new ways of approaching problems without abandoning earlier conceptions and techniques that have proved useful. The engineering metaphor of evaluation has been enormously fruitful, and in suggesting its limitations I by no means am advocating that it be scrapped. I do, however, propose that many of the evaluation problems we face can be redefined, and perhaps resolved more fully, if the problems are seen from other points of view. This chapter has suggested some alternative conceptions of evaluation problems, but it also has suggested, I hope, that an awareness of the conceptions we are using is critical for the progress of our discipline.

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Notes

1. Quoted by Howson (1973, p.166).
2. Interestingly, both systems have shown signs of breaking down under the high costs of open enrolment, and a reaction against open enrolment is being expressed.
3. In the Netherlands, a project led by A.D. deGroot (deGroot and van Naerssen, 1969) has developed an item bank for secondary school mathematics; item banks have also been developed in Australia, New Zealand and the United Kingdom.
4. Unfortunately, Freudenthal does not explain how variations in the curriculum might be handled in such studies. He appears to assume a common curriculum within countries. Although national syllabuses and examination systems exert great pressure to shape a common curriculum, they do not achieve it – especially in countries with experimental curriculum projects, various types of schools, grouping for instruction by ability, or attempts at individualized instruction.
5. For example, the Engineering Faculty of the Universidad Nacional Autonóma de México has undertaken an extensive study of its evaluation procedures and is experimenting with more frequent examinations containing some objective items.
6. See Chapter VI for a fuller discussion of the movement.

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PANEL

S. Avital (Israel), A. Bishop (United Kingdom), H. Brolin (Sweden), P. Buisson (France), E. Jacobsen (Unesco), D. Lind (Federal Republic of Germany), S. Shimada (Japan), U. Viet (Federal Republic of Germany).

In addition to the panel members, the following contributed to this chapter: D. Broomes (West Indies), G. Glaeser (France), T. Kieren (Canada), J. van Lint (Netherlands), A. Tristán López (Mexico), J.H. Lorenz (Federal Republic of Germany),

H. Radatz (Federal Republic of Germany), L. Steffe (United States), J. Timmer (Netherlands), S. Wagner (United States).

Chapter IX

Overall goals and objectives for mathematical education

Ubiratan D'Ambrosio

1 INTRODUCTION

1.1 Difficulty of the subject

Although presenting itself as a basic issue in dealing with mathematical education, the analysis and understanding of the many aspects of goals and objectives closely relate to several fields which are of little apparent relation to mathematics itself, but which cannot be ignored. This chapter is placed at the rather undefined boundary of many fields: mathematics, education, history, sociology, and politics. Although recognizing the probable emergence of the new field of didactics of mathematics, as the term is increasingly used, and accepting that it may eventually have a "status similar to that of analysis or topology" (Howson, 1973), this chapter is placed within the somewhat more urgent and challenging context of the subtitle originally given to it: "Why teach mathematics?" In fact, this is the basic issue facing teachers and children, who invest much of their time in the process of learning mathematics, parents who hope schools are participating in the education of their children, and legislators, administrators and politicians who have to choose options, rather dramatic options in the case of developing countries, when allocating resources for education.

In preparing the initial draft of the paper, we were faced with the rather difficult choice of how to approach the issue of goals and objectives. Should we, in a stratified way, try to find a taxonomy which might be helpful when looking into the problem of goals? Matts Håstad, member of the panel for our theme, was in favour of this and has suggested: "(i) skill in numerical and algebraic calculation; (ii) concept understanding; (iii) knowledge of methods used in mathematics; (iv) problem solving ability; (v) ability to apply mathematics". Or should we try to look into the theme in a global way, closer to an analysis of what Bent Christiansen has called "mathematical education as a process"? We faced the difficulties well identified by Christiansen at the end of his paper (1976b): whether to follow a societal aspect, looking into mathematical education as "a process of development imbedded in the process of development of the educational system as a whole", or to follow a systematic approach which is "under constant international debate".

This chapter is part of a publication with a broad international circulation; it therefore seemed to the author that it would be of limited importance to opt for a systematic approach related to the emerging field of the didactics of mathematics. Instead, we opted for an approach which would raise many issues, which would challenge mathematical educators all over the world to face the basic question of "Why teach mathematics?", and thus serve as a preliminary and fundamental preamble to the difficult and certainly socially-bound question of the "Overall Goals and Objectives of Mathematical Education". In conceding to this socially oriented approach, this chapter reflects the positive reactions to the first draft. Some reviewers urged the need for this kind of approach, indicating a growing concern for the place of mathematics in the societies of today and tomorrow. This growing concern, witnessed at the Fourth Inter-American Conference of Mathematical Education, Caracas, 1975, and at the First Pan-African Congress on Mathematics, Rabat, 1976, was clearly present at the International Congress of Mathematicians held in Vancouver, 1974, as well as at the Third International Congress of Mathematical Education held in Karlsruhe, 1976. It had to be recognized as a trend.

1.2 The structure of the chapter

The chapter, due to obvious limitations of space, can only be an introduction to the theme. It is meant mainly to raise issues which are not commonly emphasized when stating goals and objectives of mathematical education and to give references to sources where more traditional approaches can be found. It is hoped that it will generate fruitful concern among mathematics educators and will lead to more understanding of the responsibilities and, above all, of the power we have, as educators, in shaping the future.

This chapter has three other sections, besides this introduction and the bibliography, dealing respectively with *an historical excursion*, in which is set the tone for the remaining sections, *the role and nature of mathematics*, in which some of the more recent views on the nature of mathematics and where it is heading are discussed, and *mathematics in school*, where some strategies are discussed.

This choice of structure reflects the author's viewpoint. As already indicated, some of my colleagues urged for a more stratified treatment. They felt, and I cannot but agree, that there are very clearly identified issues when talking about objectives and goals for mathematical education. The issues are of a varied nature and although an intersection is found among the dozens of listings which we had a chance of consulting, the risk of adopting one or another is high. However, some remarks about these issues are necessary for our description of the structure of this chapter.

It is easy to identify a few questions which are invariably posed: why, what and how? This is the title of the British contribution to Section B3 of the Karlsruhe Congress at which the theme of this chapter was discussed (Blakeley, 1976), and the questions have been asked repeatedly (D'Ambrosio, 1957). They are the permanent questions of the entire field of mathematical education, and form the underlying issue of this volume. In other words, the three questions summarize all our quest, involving students, teachers, parents, administrators, legislators, etc.

We have to invite all those concerned to face the issue as honestly as we can. This has to be done broadly and repeatedly; and the answers will certainly have a dynamic character, changing geographically and with time. Some overlappings and even some apparent contradictions which may be noticed are part of the very dynamic aspect of the issue. This chapter places the issue in an overall context, with emphasis on the first of the three questions.

After listing eight commonly given answers to the question of "Why teach mathematics to most children?", Geoffrey Matthews (1976) summarizes his views saying that "the clear objective of the teacher of young children is therefore to provide appropriate experiences so that they can be helped forward in their development". In the same vein, the French contribution to section B3 of the Karlsruhe Congress, as presented by Michel de Cointet, says that mathematics should be a discipline of awakening, listing three kinds of objectives: (i) to give knowledge of the basic mathematical entities and their basic techniques; (ii) to make children discover the power of mathematics in other sciences and in the understanding of the technical and social world; (iii) to generate fundamental steps of scientific activity. We could certainly go on indefinitely with this exercise of listing accepted and agreed-upon objectives for mathematical education.

In all the listed objectives and stated goals, four kinds of components are identified: mathematical, sociological, psychological, and affective. There has been a shift towards the last three in the course of this century. It is a question whether the reasons for this shift are linked to the changes of mathematics itself, or rather to the profound changes through which society is going. Section 2 looks into this.

Certainly mathematics is changing fast. The introduction of computers, the growing interrelationship between mathematics and other disciplines, and consequently the expectations, resulting from a rapidly changing technology, of mathematical research, strongly affect mathematical education. The extent of this influence is easily detected in the enormous changes witnessed in the 1950s and 1960s, with the bitter reaction the changes produced. To what extent mathematics as it is done by mathematicians and currently applied can be brought into schools is yet to be seen. The underlying question of usefulness of mathematics as a tool against mathematics as a formative discipline, to a certain extent putting into conflict the sociological and affective components, is again a basic issue in mathematical education. It is certainly a major issue facing those concerned with the allocation of resources for education, and it is an issue of gigantic proportions for those responsible for building up educational systems, as in developing countries. No indisputable evidence for any of the proposed answers can be claimed. Simply, the issue has to be raised, discussed, and discussed over and over again; and this will probably go on forever, as long as our society and our science go through changes. Section 3 discusses these issues.

Mathematics is being taught all over the world, to practically every child of school age. What kind of mathematics should be taught? Something reflecting a traditional pattern, built up and set up under different aspirations, in a completely different social pattern? Or something carrying in its very texture this component of change and flexibility which will allow for less painful ongoing adaptations? This attitude, reflected in the growing importance of the psychological and affective components, requires less traditional goals, and new and active strategies. Section 4 discusses these issues.

1.3 The way it was written

It is important to explain how much input there is in the chapter, other than from the author himself. Although the pattern is the same as for the other chapters, the nature of the subject called for a broader participation in the process of sounding views on the issue. The first outline of the chapter was drafted in September 1975, and revised in December 1975. Fortunately, there was an opportunity of discussing the issue at length during the Fourth Inter-American Conference on Mathematical Education and of incorporating in the chapter many of the trends observed during the congress and summarized in the Proceedings of the Congress (Caracas, 1975). Three other meetings in 1976 allowed for a broader discussion of the issues involved as well as of the general structure and emphasis in the chapter: the 82nd Annual Meeting of the American Mathematical Society, held in San Antonio, Texas, in January 1976; the 54th Annual Meeting of the National Council of Teachers of Mathematics, Atlanta, Georgia, April 1976; and the First

Pan-African Congress of Mathematics, Rabat, July 1976. Apart from these, there were local meetings, some specially designed to discuss the issues of objectives and goals. In all of them, it was possible to broaden the views under study. But most important of all were the discussions held during the Third ICME in Karlsruhe, August 1976.

Besides this direct input, we must mention the collaboration received through correspondence. A preliminary draft of the chapter was sent not only to the members of the International Advisory Group described in the introduction to this volume, but also to a large number of other individuals all over the world; among them members of the Executive Committee of the International Mathematical Union, members of ICMI's Executive Committee, well known mathematicians and less known younger mathematicians, both from developing and developed countries, physicists, engineers, and other scientists, as well as specialists in the humanities. The draft was also sent to professionals from various non-academic fields, since it is our feeling that the "man in the street", who once in his lifetime went to a school and was taught some mathematics, has something to say about what he got out of this, and how he regards what his children are being taught in school. The author is thankful for the answers received, but the geographical distribution of the reactions was somewhat disappointing, showing much less concern among individuals from countries where education is such a vital issue, both from the point of view of expectations of the educational system and also from the enormous share of the national economy there involved, than from individuals from countries where it is a matter of merely improving a system that undoubtedly has served and still serves its purposes. This pattern of responses suggests a mimetic attitude to education in developing countries, where reforms and trends are sometimes adopted from elsewhere without adequate criticism and without sense of national priorities and values.

1.4 The sociological setting

This chapter was initially designed to focus on the question of the sub-title from the point of view of the development of society, of school systems and of teacher education, as well as on the changing views of the nature and role of mathematics, and on the nature of the teaching-learning process. The intention was to discuss the role of mathematics in relation both to general educational goals and also to the goals and objectives for mathematical education. These topics should be considered within a general context, including their place in different types of societies at various stages of development.

When the first reactions to the outline were received, it became clear that the ambitious scope of the topics to be covered and the depth with which we intended to cover them had to be moderated. Undoubtedly, the course we had proposed to follow would have required much more time than we were able to devote to the project, and much more space than available to this chapter. The scope of the chapter was moderated without changing its structure, hoping that some of the topics and issues in it will provoke further work on the subject. Ideally, this chapter should originate some essays on the many points raised which, we believe, deserve further analysis.

Our presentation departs from some of the traditional approaches to goals for mathematical education. Thus we shall prefer a rather casual formulation of goals in common sense terms. As an example we quote the listing given by Stephen Krulik and Ingrid B. Weise (1975) which is quite comprehensive, and summarizes what is generally agreed upon:

Goal 1 : To achieve for each individual the mathematical competence appropriate for him.

Goal 2 : To prepare each individual for adult life, recognising that some students will require more mathematical instruction than others.

Goal 3 : To foster an appreciation of the fundamental usefulness of mathematics in our society, particularly with reference to understanding and improving man's environment.

Goal 4 : To develop proficiency in using mathematical models to solve problems.

We found this listing of goals much more uncompromising than many others, and borrowing words from A. N. Whitehead, free of any tendency to exhibit pedantry and routine. This listing also demonstrates the close relationship between goals of mathematics education and general educational goals, and it points to the needs of society. Accordingly, it fits much of the philosophy underlying the Faure report (1972).

This chapter presents an analysis of the place and role of mathematics in societies of the present and of the future – and of the obvious implications for mathematical education. It could be entitled "Why teach mathematics?", thus responding to the subtitle of our theme at the Karlsruhe Congress. The importance given by Professor K. Kuratowski (1975), in his account of mathematics in Poland to the moods and tendencies of Polish society as a factor to be appreciated in the birth of Polish school of mathematics, encouraged us to look into this kind of sociomathematics, which might be thought of as a new discipline, as is sociobiology or sociolinguistics. We recognize a total interaction between science, in particular mathematics, and society. Mathematics, though it is less immediately apparent than in the case of the natural sciences, affects in an ever-increasing degree the whole pattern of thought, culture and politics. The lucid introductory chapter on "The Emergence and Character of Science" of Bernal's *Science in History* very clearly illustrates instances of this (Bernal, 1969).

This chapter has as its purpose an overall analysis of objectives and goals in the various countries and regions of the world, identifying and possibly interpreting trends. It deals with the fundamental question of "Why teach mathematics?" in the context of a rapidly changing educational pattern all over the world, mainly due to the rapid concretization of the ideal of mass education. It is universally accepted that the benefits of education, not only in the elementary grades but also in professional and scholastic fields, both in formal schooling and in some form of continuing education, should extend to the lower economic and social strata of society. It is generally accepted that literally everybody has the right of access to education, to the highest possibilities of the individual's own capacity. It is also a universally accepted ideal that society should be constructed in such a way that only a minimum of educational disparity is accepted. This conflict between what Johan Galtung calls educational growth and educational disparity leads to serious questioning about the value of general education, or mass education, as it is commonly taking form. This questioning is seen in the growing movement called "Survival" and in its various similar movements and ramifications (Joubert, 1975). But much more influential and of more immediate consequence is a new pattern of educational reform which is seen now in several countries, and which may be characterized by the idea that we should not try to obtain for the beginner the goals we envisage for higher education. We see the ideal of such a mass education in developing countries well expressed by Ahmad H. Ubeid (1975): "It is our task in education to provide every young person with, at any rate, sufficient instruction to enable him to embark in life in society without too much difficulty. He must be given an education that will help him to determine his needs and *to participate in his own instruction*." (Our italics.)

The reconciliation of three different philosophies of education – (i) the search for values; (ii) the pursuit of new knowledge; (iii) the support of a designated social structure – and the conflicts that result from them, permeate the entire study of objectives and goals for mathematical education. The search for a balanced approach is crucial in developed and industrialized countries, with somewhat stable structure, as is so well exemplified by the *Report of the Carnegie Commission on Higher Education* and the criticism it raised.

If the problem of defining priorities in education and in science poses such major problems to developed nations, we can easily understand the gigantic dimensions of the problems posed to educators, adminstrators and governments of developing countries, when allocating badly needed resources to education and scientific development. While there is a major convergence of viewpoints regarding the values and priority of free and non-directed research in developed countries, with great emphasis on university research, its priority for developing countries is more and more often being questioned (Moravcsik and Ziman, 1975). Undoubtedly, this is such a major decision affecting governments of emerging countries that no attempt other than the opening up of diversified educational experience should be accepted.

Indeed, developing countries have a much better possibility of achieving adequacy and excellence in their educational and scientific structures as a result of the competition of diversified options than they would have by following a unique approach. The well known incongruencies, difficulties and inadequacies of technological transfer are enormously greater when we talk about the transfer of educational models, including models of the structure of scientific research. It is a major premise that education, in particular science and mathematics education, hence scientific research structure, is a socio-cultural phenomenon. Its structure, and hence the teaching models, curricula and evaluation schemes are all products of socio-economic and cultural constraints, guided by national priorities and goals. They are less dependent on the intrinsic structure of science itself and – for any given society – even less dependent still on the philosophical and psychological framework of foreign societies.

On the other hand, the political and economic structure of the world and the absolute interdependence of the various countries, nations and societies, with the resulting competition, makes it absolutely essential for emerging countries to close the so called technological gap between them and developed and industrialized countries. In order to be incorporated in the concert of free and independent nations, and to relate economically and culturally with dignity to bigger countries, emerging developing countries need a sound and capable intelligentsia. To reconcile these goals, that is to enter into a dignified and honest competition with other countries whilst at the same time preserving traditional socio-cultural and moral values, seems to be a major problem facing educators in all countries, both developed and developing, rich and poor, small and large. Within this context are located science, in particular mathematics, and education. Within these overall goals, the theme of the chapter is discussed.

2 AN HISTORICAL EXCURSION

2.1 The acquisition of numeracy

We distinguish several characteristics of education which may serve as a guide for a brief historical account of it. We distinguish *a purely vital education*, instinctive, through which children learn how to survive and to continue the species, which has been traditionally put under the responsibility of families or structures which remind us of a family environment, and *a societal education* in which children are taught basic attitudes and conduct and acquire the first moral values, which was also traditionally placed in the hands of a family structure or the like. Maternal and child care centres we regard as educational structures of a family type. These are the basic components of what Margaret Mead would call a post-figurative model (Mead, 1970), which we find present in practically all educational structures. Then we distinguish two other forms of education, which we agree to call *artisanal or professional* and *contemplative or speculative*. In the first category we place Greek, mainly Spartan, and Roman models, as well as the Chinese public service structure, all the civilizations of antiquity, and all the initiation cultures which prevail to the present day, mainly in Africa. At the same time, we notice an intermingling of the second category, represented firstly by religious practices and motivated by understanding the place of man in the world and clearly the product of a certain economic and social stability in its first manifestations. The sophists illustrate this, and so to a certain extent does Roman court life. Certainly in this intermingling we can identify the genesis of scientific ideas. In this somewhat nebulous beginning, we can identify instances in which the two structures of education join forces and new knowledge is created. On the other hand, we can distinguish the preservation of the two structures in the roots of a society of classes, as it was so clearly put by Mattoso Camara (1975) in his studies of the history of linguistics.

The beginnings of mathematics are, in the same vein, part of a context, but it is probably an easier task to illustrate through mathematics this link between the two educational structures we mentioned above. The very beginnings of both number words and number symbols show strong practical components as well as linguistics features which are a reflection of the contemplative aspect, as for example the one-two-many, singular-dual-plural in Greek grammar, placing the step to three as "a step across the threshold of darkness" (Menninger, 1969). In the same vein, the work of Claudia Zaslavsky (1973) illustrates a definitive pattern of practical considerations, in African cultures, which evolved to other forms of considerations. As R. L. Wilder (1968) puts it, the most likely cultural component we will find among all intelligent and culture-building beings is the existence of counting processes, and this may be an argument in the context of a contemplative or speculative scheme.

It would be worth studying comparatively the development of mathematics in civilizations where the abacus was commonly used. Although taken for granted as part of theoretical mathematics in western cultures, and somewhat misplaced in medieval schools, ordinary arithmetic (by which is meant the skill of performing calculations) is nothing more than a mechanical device, and hence not to be considered as the core of mathematical studies as it has been since medieval times. In this respect, the argument of Rikitaro Fujisawa, when discussing the teaching of arithmetic in elementary schools in Japan, is interesting. This was published in 1912 in the *Summary Report on the Teaching of Mathematics in Japan*, appearing as an Appendix to the important treatise of Yoshio Mikami (1974) on mathematics in China and Japan.

In fact, while only little emphasis was laid upon calculation by Greek mathematicians, this was not the case with Roman civilization. The enormous emphasis given to number theory by the Pythagoreans and others, which was carried into medieval schools, has to be distinguished from ability to calculate. As Plato put it in Book VII of the *Republic*: "For in dealing with encampments and the occupation of strong places . . . a slight modicum of geometry and calculation would suffice. What we have to consider is whether the greater and more advanced part of it tends to facilitate the apprehension of the idea of good". Also in Aristotle, who attributed to numbers a very physical meaning for reasons clearly discussed in Kline (1972), we notice a minor, almost non-mathematical, role for computation – and no place for it in school.

2.2 The Middle Ages and the Industrial Revolution

The Greeks transmitted to us unequally developed branches of mathematics, – a deductive and systematic geometry, with considerable insights into number theory, and an undeveloped arithmetic, heuristic and empirical and based essentially on calculating devices not considered part of mathematics. The latter branch was pursued by the Romans, who made practical use of measuring and counting, developing several forms of abaci and finger counting. This technique, which according to Menninger is a true calculating machine, "seems to have been passed on mostly by word of mouth. No textbook describing it has been preserved from Roman times. This very fact may be the evidence of its widespread use among common or illiterate people, . . . which do not require special teachers or schools". In fact, what was the domain of artisanal or professional education entered and dominated education at a moment when opportunity for education began to spread, when universities were being founded and technology was laying the grounds for the Industrial Revolution. And this happened mainly because of the introduction of the experimental method, and consequently of a quantitative approach, as the new method of the natural sciences. The new social forces of the Renaissance and Reformation combined to make possible a major change in ideas in science and replaced the whole structure of science inherited from the Greeks by a new quantitative, atomic, infinite and secular view of the world.

Modern science gained momentum with the development of a new economic order towards the end of the Middle Ages. The growing importance of towns, trades and industry forced a deviation from the economy of feudalism. The bourgeoisie, which appeared in previous centuries, imposed a form of payment which replaced, by the fifteenth century, forced service as a form of payment, and which imposed itself finally in the middle of the nineteenth century in the whole of Europe. At the same time, this period laid the foundation for the Industrial Revolution, which was strongly dependent on the rise of the new science, experimental and quantified. And at the same time, another component was brought into play, the opening of the frontiers of the world through navigation and conquest. This in turn facilitated the expansion of technology, by introducing new and cheap raw materials into an economy which was increasingly geared to the processing of those materials.

In this period, the Middle Ages through to the nineteenth century, schools went through radical changes. While we see in the Middle Ages the appearance of universities, we also see the development of what we have called above artisanal or professional education, resulting from a form of professional guild, in a sort of parallel structure. At the end of the Middle Ages the discoveries of new lands, the invention of the press and a new socio-economic structure strongly influenced by the discoveries, strengthened the position of the universities and enhanced public elementary and secondary education.

The place of mathematics in education in this period was quite weak. When Richard Mulcaster published his *Positions* in 1581, he suggested that education should take into account the "natural" development of children, and that teaching should be restricted to reading, writing, drawing and music, much in the Aristotelian tradition. While artisanal schooling was aware of the progress underlying the scientific and technological revolution under way, elementary schools preserved their traditional appearance, based on Greek learning. Modern pedagogy owes much to the ideas advanced by Comenius in his *Magna Didactica* of 1633, which set the basis for a new and more practical schooling. At the same time, schools for the nobility also acquired a tendency to professional training. The accounts of early education in America, collected in the *32nd Yearbook of the National Council of Teachers of Mathematics of the United States*, give a good picture of what was going on in education in Europe and in particular of mathematics in the context of the new educational order. A most popular book was Dilworth's *Schoolmaster's Assistant*, published as early as the mid-sixteenth century in England. Arithmetic there appeared as the art of computing with numbers. Whilst in Europe a general education did not attach much importance to mathematics and was reluctant to adopt the concept of a practical education for all, this idea prevailed in America and mathematics was incorporated as a practical tool. Nonetheless, mathematics was given less importance, irregularly taught, and often given by a private tutor. But its character was quite different from the scholastic mathematics of the Greek tradition. We refer the reader to the cited yearbook as a rich source of information on the early steps towards the establishment of mathematics as an independent subject in America, which very much reflected the most advanced trends in Europe.

2.3 The twentieth century

We thus reach the end of the nineteenth century and enter the twentieth century, which showed strong motivation for mathematical research, deriving from many sources.

Paradoxically, while there was an excess of conservatism in both mathematics and education in the first half of the century, there was a profound richness in the new directions which science and society were taking. We are now living through a second scientific revolution, with new fields of research being opened up, with new tools for the exploration of nature, both in smaller and larger dimensions than man's imagination can follow. Fields hitherto considered by many of the realms of religion are beginning to be established as scientific disciplines, as for example, parapsychology. Techniques hitherto appealing to millions of readers as science fiction are being investigated in laboratories, as for example the creation of life. In a more routine vein, organ transplants are today almost commonplace, the huge computers of the 1940s have been reduced in size to a cigarette pack, and the "death rays" of last century fiction writers, dramatized by H. G. Wells in his *The War of the Worlds*, are now a reality, fortunately being used for instant internal surgery.

In the same way society witnessed profound changes in the first half century. The events of the period, some within the span of our own experience, show a transition from one kind of society to another, from one economic order to another, and an ever increasing interaction of science and society. The balance of power that resulted in the challenge of the capitalistic system, entrenched for 300 years as the economic model for a world under European dominance, by the socialist system, which has arisen out of the internal conflicts of capitalism itself, facilitated the recognition that the majority of the world population lives under conditions which were barely imaginable at the turn of the century. From the end of the last century the foreign and colonial policies of the big powers have been largely dictated by the urge to secure great shares of the world markets for the products of the growing national industrial trusts. Largely favoured by the balance of power which grew out of two world wars, newly independent nations entered the picture of world economic equilibrium, increasingly playing a more important role. Having achieved political independence, these nations are struggling for economic and cultural independence, and search for their somewhat forgotten moral and cultural identity. It is unnecessary to recall the changes in the world economic pattern resulting from this new political picture. The same goes for the attitude and values of society as a whole, with enormous implications for schooling.

We close this section within the present day framework of profound social, political, economic and behavioural changes, interacting with not less profound changes in science and technology, and the resulting need to question the moral values which necessarily play a role in this interaction. Mathematics and mathematical education are part of this complex.

3 THE ROLE AND NATURE OF MATHEMATICS

3.1 What is mathematics?

Mathematicians and some non-mathematicians can identify with ease mathematics in intellectual experiences, and can distinguish mathematical reasoning and method in practically any situation where they are either present or potential. I cannot but agree with the position of René Thom (1975) in placing mathematical thinking as a finer language to simulate external phenomena than usual languages. This places mathematics within the ample context of scientific reasoning and method, demanding an analysis of its role and nature to place it in a general context and in its relation with other languages which try to simulate a reality. Indeed, the process of mathematical creation proceeds like any form of creation, as the result of some internal force which we cannot identify, while the object of mathematics is not the result of a purely sensorial experience, a pure observation, but the combination of this with some process of abstraction and correlation of ideas. This is what Thom (1974) calls "something like an alternation of reductionist and structuralist tendencies" (Trans.). Having assumed that mathematics is itself a morphologic discipline, since it studies forms and structures which appear as given, the reductionist and structuralist tendencies are basically related to what may be called motivation, and to what can be distinguished as consolidation or systematization of knowledge.

Other attempts to explain how mathematics progresses seem to agree that there is a peculiar state of mind which leads to the creation of new mathematics. To describe this peculiar form of creativity is very difficult. An important source for the analysis of scientific creativity is provided by scientists themselves. We have a few autobiographical essays written by mathematicians, and I would urge more such essays to be written. Not only this, but more studies of mathematicians are needed. Fortunately, there are a growing number of books and articles in this line.

3.2 Interdisciplinarity and creativity

If we look to the future of mathematics and try to identify trends, it is very difficult. A major force in shaping the future of mathematics is its growing relationship with other scientific fields. At a symposium held in 1971 during the 138th Meeting of the American Association for the Advancement of Science, Harold Grad stated that all mathematics has been or can be applied constructively to some natural phenomenon, and that probably the major road block to actual applications of inherently applicable mathematics is the incredibly obscure way in which mathematics is conventionally presented. He also claimed that a person's interest will determine the direction in which he will go. We entirely agree that this is the case and consequently concentrate much of our considerations about the future of mathematics around its relations with other disciplines and its marked trend towards interdisciplinarity.

An analysis of priority areas for research in the United States shows a growing trend towards interdisciplinary research. As Peter Hilton – member of the panel for our theme – remarked, reacting to our comments that there is a need in developing countries to "qualify" mathematics research: it is the responsibility of mathematicians to direct the attention of their students to the important areas of applications of mathematics, and also to ensure that their students understand the methodology of applying mathematics, although there may be a danger in too much interdisciplinarity at the undergraduate level. What may be seen as major factors determining the future of mathematics research are those factors which may be called "external" but which will certainly shape priorities in mathematical research. The references made by social scientists to catastrophe theory, in which they accept that it will provide an effective tool for the study of their discipline, are very significant. There is no doubt about the recognition of a trend towards a new relationship between mathematicians and non-mathematicians. This changing attitude, which may be quite stimulating, is not a matter of choice. As a consequence, new foci of interest for research mathematicians are coming through an osmotic process. This process goes both ways and the appearance of completely new scientific fields, bearing characteristics of mathematics as we know it today mixed with those of other disciplines, is quite possible. The role of computers in such new scientific disciplines will probably be a major one.

Is this a future that is within the reach of our life time? Probably so; but certainly it is in the life span of the children in our schools. The implications of the above considerations for schooling are of considerable magnitude and may strongly affect mathematical education. The emphasis should be changed from what is taught, that is, from curricula, syllabi and contents, to methodology, and to a new classroom environment, both geared towards creativity in its fullest sense. We have to accept the dangers, pointed out by Peter Hilton, of this shift of emphasis away from content. Certainly, this is an area where further involvement by mathematicians is needed. Something must be said on the process of discovery and creativity. This entire change was magnificently captured by Professor H. Freudenthal when he replaced the famous principle of Comenius's didactics: "The best way to teach an activity is to show it" by his own principle "The best best way to learn an activity is to perform it". Indeed, the focus in schooling should be on the learner. In particular, mathematical creativity may be gained throughout school years if mathematics is done as activities in those years. Creativeness may form the core of mathematical education, but has a much broader extent. It is by no means restricted to mathematics. It is a global attitude, hopefully enhanced by mathematics. This is very well put by J. A. Dunn (1973) in his review paper, which contains an excellent bibliography on mathematical creativity. As Dunn says, in mathematical activity in the classroom the subject matter is known, while mathematical activity in professional life is like a creative art. While recognizing the importance of learning theory for the considerations relevant to this section, I will not touch upon this subject, referring to Chapter X and to the excellent bibliography assembled in Dunn (1973 and 1975).

I could hardly claim that the observations made above represent any more than my personal view of the subject, supported by isolated remarks from other speakers in the section. My perception of the direction mathematical research is taking fits ideally with what I see as a suitable scientific and technological structure for developing countries – dynamic and self-modifying to serve the constantly changing needs of society. This scientific and technological structure goes side by side with an educational structure which must have the same dynamism and power to reform itself during the process, to fit the expectations of the society in which it is placed. These expectations will be examined in the next section.

4 MATHEMATICS IN SCHOOL

4.1 Why teach mathematics?

Here we reach the core of the chapter, and we look into the question of "Why teach mathematics?" in the framework built in the previous sections. Bent Christiansen (1976a), in his outline of major developments in mathematical education in the period 1960–76, mentions as categories of the present goals of mathematical education in individual countries those related to the nature and role of mathematics, to the learner's working methods and to his use of mathematical methods and principles, and to knowledge and skills. In these broad categories, many of the specific goals can be inserted. We would prefer not to categorize goals, but look into the subject in a global way, emphasizing that "our primary concern should be the improvement of the overall education of the pupil – that the teacher's job is to educate and not merely to instruct", in the words of Griffiths and Howson (1974). Of course, there is no contradiction between this approach and the categorized way of looking into goals, but we can get more flexibility and dynamism in education, mainly in mathematical education, if we look into the matter in a global setting. Again, there is no contradiction with a more stratified approach, as for example the one presented by W. Servais in a contribution related to the work of Section B3 of the Karlsruhe Congress. We find there a listing of sixteen items categorized as general objectives for building up an intellectual, aesthetic and moral background. Among these we may mention as examples: "to acquire a constructive critical sense" and "to activate a taste in the expressions of thought: clearness, order, conciseness, elegance" (Trans.). We would rather place these as consequences, as subproducts, as natural attitudes within a context, and not as objectives, which *a* *fortiori* imply more immediate aims. Certainly, mathematical education is not the only route to achieve this kind of behaviour which, in the order of values upon which western society traditionally rests, may be regarded as an ideal of excellence; and probably it may not be the best route available.

Indeed, we notice some changing attitudes towards mathematics holding an undisputed place in the educational context. As we read in the NACOME Report (Hill, 1975) "current cultural preoccupations no longer award mathematical, scientific and technological disciplines the first place of honor (and funding) that has been the case for many years". I do not see much point in trying to defend the place of mathematics, as the same report does when it says that "Whatever alternative fields of scholarship take precedence over mathematics . . ., our world remains a scientific, technological, and industrial world and mathematics remains essential to its caretaking and improvement". This in itself represents an objective for mathematical education which is, obviously, open to a deeper and more menacing criticism. As is recognized in the same report, education has been conditioned by recent cultural changes which deeply challenge the scientific, technological and industrial world within which many of the objectives for mathematical education arise. I see this process of making our societies more and more based on advanced science and technology as an irreversible one, and indeed a favourable one. There is no point in rejecting scientific and technological progress on the basis of some bad uses which have been made of them.

Hence, as a premise to the continuation of the discussion of this section, it must be made clear what kind of societies we are talking about when trying to locate mathematical education within these societies. We must accept Utopia as our ideal society, which I feel quite legitimate for us to think about when discussing the future, and education leads to this future. It is irrelevant to discuss details of Utopia. No one will disagree that this Utopia, in many of its several forms, includes justice, freedom, high quality of life, etc. Towards Utopia humanity has been building up the various kinds of societies, and has been accumulating knowledge and technology and destroying myths. In this context, education has been evolving. Mathematical education is placed in this context.

4.2 A utilitarian view

There is a growing need of trained mathematicians for applications and for the use of technology in its various stages. Examples can be found in most countries which show the concern of mathematicians about the distance between what is taught or researched in mathematics and what is applied. This growing concern and growing emphasis on applications is well expressed by J. J. Seidel (1973) when referring to the mathematical engineering curricula in the Netherlands: "Applied Mathematics was (before 1950) mainly developed by representatives from other disciplines, such as physics and engineering. However, in the last 15 years there has been a considerable change". Whether accepting the defensive posture of Peter Hilton (1975) that "Applied mathematics is not an alternative to pure mathematics; it is the superimposition of an extra and very difficult skill on top of the already difficult problem of learning mathematics", or his much milder "doing applied mathematics is doing mathematics within a context of concern for the solution of scientific, non-mathematical problems", or accepting the agressive position of Morris Kline (1972) in saying that "Mathematics is a simple subject compared to economics, psychology or physics. It is a narrow subject . . . Moreover, mathematics *per se* does not deal with human beings and the complex problems that dealing with human beings pose", we have to recognize that society at large expects return from mathematics.

This digression has brought us to the long and sterile dispute about pure versus applied mathematics or its variant, mathematics versus applications. As Chandler Davis points out, it is wrong to assume that the improvement of relations between the "mathematical and the outside non-mathematical world" depends on changing the ratio of pure and applied mathematics or of mathematics and its applications in the schools. It is mainly a matter of attitude, which has to be generated in early schooling.

At this point, we may claim that society on the whole expects mathematicians to be competent professionals, capable of justifying why they are being paid to do mathematics. This is a very crude, unsophisticated way of putting it. Since the 1950s research mathematicians as a class have enjoyed a prestigious position, without precedent, with the expectation that much, if not all, of what was being done on mathematical research would, in an almost immediate future, be translated into technological progress and development. Practically no account of what was going on in mathematical research was ever asked for.

Mathematical education is a reflection of the position mathematics and mathematicians enjoy in society. While providing speculative or contemplative education directed to an elite, leaving professional training to a parallel structure, the expectation of society regarding education was much more directed to individual fulfilment. With mass education and the incorporation of professional training into the educational system society expects much more. We notice in most of the lists of objectives a strong emphasis of societal behaviour and reaction to the external world. The *Association des Professeurs de l'Enseignement Public* of France, through Michel de Cointet and Michèle Couchan, lists, with great emphasis, the development of attitudes as a major objective of mathematical education. The student should be able to, as an objective of his schooling: (1) analyse different components of a situation; (2) recognize analogous situations; (3) choose a strategy adapted to a situation; (4) make himself understood by others; (5) have a critical attitude; (6) construct simple deductions and construct a chain of deductions; (7) predict a result and generalize; (8) construct a simple model; (9) participate in a collective task.

Very similar are the objectives listed in the curriculum guide prepared for the Secretary of Education of the State of Sao Paulo, in Brazil: (1) develop the capability of analyzing, relating, comparing, classifying, ordering, synthetizing, evaluating, abstracting, generalizing, creating; (2) develop the habit of studying, of rigour and precision, of order and clearness, of correct use of language, of conciseness, of endurance in obtaining solutions for problems attacked and for criticism and discussion of results obtained; (3) acquire specific abilities for measuring and comparing measures, calculating, constructing and consulting tables, tracing and interpreting graphs, utilizing and interpreting correctly the symbolism and terminology of mathematics; (4) acquire information and knowledge about several concepts and methods utilized in mathematics; (5) develop the capability of obtaining, from given conditions, valid results in new situations, utilizing deductive methods; (6) recognize the interrelation among the several fields of mathematics.

With minor variations, this is the pattern of listed objectives we have found all over the world. Whatever the set of objectives, questions like "Why?", "What?" and "How?" are present in any effort to build a better curriculum. In the context of developing countries, the major efforts to bring about alternative solutions based on a local answer to the classical three questions are, in many instances, more successful than foreign ideas and models brought into the country without the appropriate criticism and adequacy.

4.3 A speculative view

The professional or utilitarian aspect of mathematical education has been discussed. Let us turn to a general cultural framework, aiming at the understanding and mastering of the world, in a speculative way. The first aspect, the utilitarian vein, probably much more affected by the introduction of calculating machines and computers, goes much in the direction of a training. This has probably to be associated with professional schools, and the main objective is to develop skills. Again, the aim in this case is a direct approach to applications, to reality, in a most unsophisticated way. The aim is to provide skills in calculating, which is probably done in a much better way with hand-held calculators and, according to specific needs, with more advanced techniques.

While this practical objective of mathematical education was based in mass education, it was confounded with other objectives, and in fact assumed the place of mathematical education par excellence. The so-called "modern mathematics" was to a large extent a reaction to this one-sidedness of mathematical education. Unfortunately, new mathematics came to replace "practical" mathematics, and what we see today is a struggle to make one approach to mathematical education serve two absolutely different objectives.

The second kind of mathematical education, which was displaced after the industrial revolutions and could not find its way back to the general context of science education, is part of the effort to develop an inquisitive, creative, free individual, fulfilling the aim of education as "the acquisition of the art of the utilisation of knowledge" (Whitehead, 1948). It is necessarily related to global reality. Again, in the words of Whitehead, "The solution which I am urging is to eradicate the fatal disconnection of subjects which kills the vitality of our modern curriculum. There is only one subject-matter for education, and that is life in all its manifestations. Instead of this single unity, we offer children: Algebra, from which nothing follows; Geometry, from which nothing follows; Science, from which nothing follows; . . . The best that can be said of it is, that it is a rapid table of contents which a deity might run over in his mind while he was thinking of creating a world". The objective of this much more structured creative form of mathematical education, closer to the real problems, with lots of plausible reasoning in the line of G. Polya (1954) and the child's need to live and grow in his own way, following the approach of H. Whitney (1975), is to place mathematics again in its position of an extremely convenient and successful language to simulate the real world. The objective is to produce new mathematics, to create new theories, to help in the solution of the new problems which are only now being identified and recognized. The objective is to give mathematics a new dimension that will fit a new world and societies which are closer to each other and looking for their Utopia. For this we need new mathematics, we need all the creative power of youth, we need new forms of thought which we cannot envisage. The primary objective of mathematical education is not to perpetuate knowledge or to push existing knowledge a little further (this will go on or will fade away as a result), but to foster the creation of new knowledge. Teaching is not the essential goal of this creative or contemplative form of mathematical education – the fundamental goal is to prepare a favourable setting for the creation of new knowledge. Let us insist that this objective runs parallel to, and is in fact aided by, the practical or immediate goal of furnishing mathematical techniques and tools for professional or daily activities.

In developing countries this dual aspect of mathematical education deserves a much more explicit definition in the statements of their goals and objectives. While large and well established educational systems have mechanisms to identify what might be called "mathematical talent" and to provide, through their traditional cultural establishment, some stimulation for the pursuit of scholarship in mathematics, developing countries need special strategies.

4.4 Strategies

One of the major tasks of mathematical education is to propose strategies which allow for the parallel running of its two apparently conflicting objectives, the one based on the concept of mathematics as a utilitarian body of skills and techniques, designed and taught to satisfy social needs, and the other looking into mathematics as a component of the large body of models of thought and languages which simulate external phenomena.

In fact, a new educational structure is finding its place in the world. A major aspect of the change refers to numbers of students. We have to agree with Dragoljub Najman (1974) when he says that "There is a fundamental disparity between institutions of higher learning which had, . . . six million students and which will have tomorrow eighty million or one hundred and ten million . . . there is an enormous qualitative difference between one situation where, in the entire world, twenty years ago, there were only six hundred thousand professors in higher education and, . . . in a few years (later) close to four million" (Trans.). The same kind of remark would apply to other levels of education. As a pure consequence of numbers involved, we are closer to the concept of a learning community, a global education context. This structure fosters the creation of new knowledge, which is brought to a readily accessible form, readily adaptable to uses in service of the community, through an integrated system of universities and production centres.

Within this context, mathematics will probably find the revival of its creative power. Naturally, what is needed is a departure from structured curricula, based on contents which are decided and fixed *a priori*. In a much more dynamic approach, teaching is structured in a series of interdisciplinary activities, resting strongly on motivation from which units of contents, in various levels of rigour and variable duration, to a large extent self-contained, are generated. Very much in agreement with the concept of Whitehead of eradicating the fatal disconnection of subjects, Jean Piaget (1975) claims that "an increasingly large place must be reserved for new points of view – interdisciplinary by nature – like those which are being developed today by psycholinguistics, decision theory, economic psychology and sociology, and so on. This however does not mean more class hours . . . but rather . . . a systematic broadening of outlook". Considerable emphasis should be placed on pedagogical efforts to present advanced theories in a form which makes them accessible to audiences with various backgrounds and differing motivation. Efforts like, for example, Solomon Lefschetz's lectures on the applications of algebraic topology, or C. P. Bruter's *Topologie et Perception*, or Ian Stewart's *Galois Theory*, or Silvanus Thompson's *Calculus made easy*. As Professor Santalo (1975) puts it, "There are topics of advanced mathematics which . . . should be brought to secondary school or university teaching for non-mathematicians . . . This constitutes a major challenge for pedagogy and the didactics of mathematics. It should be aimed at teaching those subjects without intermediate steps, even at the expense of normal rigour being replaced by an intuitive approach" (Trans.). But this cannot be achieved within a previously fixed curricular context. A moderate approach to this unstructured curriculum was described by E. G. Zeeman. His approach is based on two principles: first, let the student do what he likes; second, persuade him to do at least one thing sufficiently deeply to affect him for life. Another principle which he brings to issue is that "people are more important than structure, so you get people first and then you set up the structure around them and the type of co-operation will depend on the haphazardness of the individuals you have". This form of unstructured curriculum can provide for the learning environment we propose.

On the other hand, some techniques of methodology of access to information become more and more needed. Information retrieval and the proliferation of audio-visual technology applied to education seems to be impossible to postpone. At the risk of sounding paradoxical, I claim poor countries should see modern educational technology as playing a key role in their future. In particular, much mathematics information can be placed in cassettes, both audio and video, and be made available to a practically unlimited clientele in different regions of the world, while problem posing and the motivation for using those stored units of knowledge cannot be generated other than locally. The two forms of education, which we called professional or artisanal and speculative or contemplative, follow the same pattern and reinforce each other as part of a same context. Strong emphasis on problem solving, both leading to the development of a more speculative attitude, like Polya (1954), or to a professional attitude, as for example the set of problems assembled by Ben Noble (1967), may provide a good starting point for this motivation oriented approach. As F. J. Almgren Jr. whose work on Plateau's problem gives a good example of the combination of very intuitive ideas, in a real and even experimental context, with highly sophisticated techniques, says: "a student of mathematics should come into contact with a broad range of ideas in many different fields even though at any particular time many of these may be understood only superficially. I believe that much of our mathematical thinking is by analogy and that we need as many things in our heads to make analogies with as possible. I know of no particularly good way to be very selective in this process of accumulating ideas". Certainly, the creation of an atmosphere of enquiry, interdisciplinarian and not pressured by *a priori* standardized curricula, would be much more favourable to the creative process we expect from mathematics.

There can be no doubt that this approach responds more realistically and more immediately to the needs of developing countries. A more detailed description of such a model is to be found in D'Ambrosio (1976). A larger scale experiment, akin to this model, is the project known as "CPS Bamako", a Unesco sponsored interdisciplinary graduate programme in Mali, now in its sixth year. A more general experiment, in the sense of being interdisciplinary and with the more immediate aim of generating foci of reform at the primary, secondary and undergraduate levels, is under way in Latin America, under the joint sponsorship of the Organisation of American States and the Brazilian Ministry of Education at the State University of Campinas, Brazil. With the initial goal of preparing, at a Master's degree level, 128 specialists in science and mathematical education, from all over Latin America, in a six year period, the project represents a testing ground for the motivation-based approach, with traditional contents appearing as a result of the evolution of the course.

We are convinced that a sound strategy for reform is to implement it on all levels simultaneously. As Professor Howard Fehr said, in his lecture delivered at the Fourth Inter-American Conference on Mathematics Education, December 1975, in Caracas, "The teachers who are responsible for improvement of their profession must possess an adaptivity to change . . . they must be aware of new goals of teaching brought about by new technology and the need to experiment with these new intrusions into an existing program". A realistic and dynamic approach to change means, in its essence, built-in mechanisms of change, in which teachers and learners are brought into a single organic component.

In fact, I believe this motivation-based approach does not add to existing learning theories. Lack of time to pursue this further did not allow me to elaborate on psycho-pedagogical aspects of the proposed approach. The extensive book by Erich Wittmann (1976) is certainly a guide for an analysis of the approach from the point of view of mathematics learning.

In discussing the approach with respect to primary and secondary teaching, one of the main difficulties we face in the experiments described above is a reaction to change. Professor Fehr's remark carries in it an objective of prime importance. At the university level we work with adults, conditioned to a pattern of schooling, and one of the first tasks in our experiments has been the demystification process. On more elementary levels, mainly in primary schools, this is not the case, which is in itself a favourable factor. Indeed, there is much more going on in this direction than we would be able to mention.

In conclusion, it must be emphasized that formulating a strategy for mathematics teaching should be an integral part of the definition of its objectives and goals. And these objectives and goals must necessarily be regarded as part of the targets of the society in which they are incorporated. In this context, mathematics itself is a strategy in the overall societal aspirations. Like education, mathematics is involved, together with all other forms of knowledge, in the set of strategies created and developed to reach Utopia. It is not necessarily part of Utopia.

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Chapter X

Research related to the mathematical learning process

Heinrich Bauersfeld

1 INTRODUCTORY ORIENTATION

It is in the periods of deep economic depression, as a comparison of long range economic cycles indicates, that the creative framework and fundamental research are prepared which are to build the substance of the following period of ascension and flower. If this theory is true, then we are in the middle of a highly creative and challenging period of research in mathematics education. Indeed, since the early 1970s most of the countries in the world have been suffering from increasing economic difficulties, for different reasons admittedly. In consequence, budgets for education, research grants and project fundings were cut down severely. Among others, the volume of research publications related to mathematics education, reported in *New Trends in Mathematics Teaching*, volume III (Unesco, 1972) as "increased dramatically", has diminished.

Parallel to this decline, criticism becomes harder. Administrators, politicians, curriculum developers, and teachers, who are complaining about the poor guidance that research gives to their decisions, now receive increasing support from inside the research community. "The hitherto impregnable fortress of experimental psychology", as Shulman (1970) said, "is currently being shaken by that most devastating of saboteurs – the critic from within". Stimulated by both the economic pressures and the difficulties of the usual research procedures, not only particular approaches, concepts, or methods are called into question by researchers themselves, but also fundamental research models and their underlying paradigms. The economic cycles theory can be verified in so far as there are already important new research issues as well as changes in the themes and in the general orientation of research that would have been incredible a few years ago – though there are many points of contact with older and neglected ideas. The reality of mathematics instruction, and mathematical learning particularly, has been influenced by research outcomes on a very small scale only. Research has limped after the needs of school practice rather than hurrying on ahead. Recent indications suggest a change in this deplorable state of affairs. Within the last decade research interests have switched over from the curriculum and from the pupil to the teacher. To some extent this is a reaction to the follow-up problems of the first big wave of curriculum development, but it is also an answer to the problems of generalizing from research. Accordingly, isolated curriculum theories and learning theories have been increasingly put in doubt, while the development of more complicated teaching-learning theories has been favoured. The latter consider the teacher, and moreover they are also concerned with the interactions between pupil, teacher and curriculum.

How did the change come about? Until recently, both research and development had focused on only one of two main determinants of the learning process: the pupil or the curriculum. They did not consider the influence of the teacher nor of the general context of instruction. Theory-related research has particularly focused on the pupil. Since the 1920s researchers have tried to identify stages of the pupil's learning and investigate the conditions of his success (experimental psychology), and to describe the developmental stages of his thinking (developmental psychology), aiming towards a general learning theory. School-related developmental work of the 1950s and 1960s, on the other hand, was engaged in the curriculum in a wide sense. Mainly by means of projects, pupil's books, learning aids and other media were developed, as well as an analysis of the structure of the subject matter and teaching methods related to it (see Chapter VII). Both movements were characterized by a neglect of the teacher's role and of the general context of learning. Consequently, both ran into increasing difficulties. Neither have the big curriculum projects improved mathematics instruction and mathematics learning on a broad scale, nor has research developed a valid general learning theory. Only competing explanations for partial views of learning have been developed, such as Piaget's genetic theory of epistemology, Gagné's hierarchical model, or the gestalt theories. These positions cannot be integrated; they represent generalizations of only limited value for explanation and prediction (Easley, 1976).

> Discussing Piagetian theory of mathematical learning Lunzer (1976) suggested, that "Piagetian theory asks the right kind of question, but is nevertheless inadequate both as a psychological theory . . . . . , and as a rationale for mathematical education". Specifying this statement he pointed out e.g., that Piaget's theory "fails to isolate the crucial role of the interface between the representation of physical reality and its relations on the one hand, and operations carried out on a mathematical model on the other", and more general, that it "makes too little reference to psychological process variables in general".

> Reviewing the "suggestive findings" from cross-cultural Piagetian research P. T. Ashton (1975) states that "they raise doubts about the Piagetian assumption of invariance in stage sequence" and, more fundamental, that "they challenge the existence of a generalized underlying cognitive structure" (p.480).

> Gagné, "whose writings could be suitably employed to represent educational neo-behaviorism in the earlier chapter, has now announced himself as espousing an information-processing view (Gagné, 1974)", as Shulman has described this rather dramatic change of faith (in Bauersfeld, 1976, p. 65).

An example of Ausubel's well known theory may highlight the difficulties with the experimental control of highly abstract principles, derived from such theories, and their explanatory or predictive power. In a recent review of 32 studies Barnes (1975) answered the question "Do advance organizers facilitate learning?" by saying that they "generally did not have a differential effect" (p.649). In particular, only 2 out of 8 studies directly related to mathematical learning showed significant results. Whether an operational definition of the concept of an advanced organizer (which Ausubel did not provide) or the study of long-term effects (which were rare, since 5 studies only had a length of treatment of more than 10 days) would produce better results is left open to further research (see section 2.6).

In particular, positions such as those referrred to above are not sufficient for the planning and realization of classroom practice, as school practitioners have complained. If it is at all possible, more sophisticated theories of teaching *and* learning seem to be necessary.

The analyses of the possible causes for the poor success and the different assessments of the analysis have produced quite different reactions and new approaches within the last few years. Thus the research scene has changed noticeably. This is the genuine concern of this report. For the identification of the general characteristics of relevant research within the 1970s my main criterion has been the potential relevance for changes in school practice and the conclusions derived for future research.

2 MAIN TRENDS

The first five points characterize goals and methods of research in the social sciences related to mathematics learning. The next five are then concerned with the teaching-learning process.

2.1 Investigation of interactions

An immense number of previous research studies tried to produce statements about the relationship between two variables from representative samples, e.g. mathematics achievement as related to sex or the comparison of two treatments. The correspondingly immense number of contradictory or non-significant outcomes has led to more differentiated questions such as the investigation of interactions: Does it make a difference in mathematics achievement when a male teacher or a female teacher teaches a male student (or a female student)? Or, is treatment A more effective with highly-motivated students and treatment B with poorly-motivated students, or vice versa? Data about such interactions can give better hints for matching treatments and groups of students, though for limited situations and for relatively elementary content only.

> Reviewing aptitude-treatment-interaction studies and drawing conclusions concerning the issues of method and generalization in research Cronbach (1975 and 1976) stated: "A general statement can be highly accurate only if it specifies interactive effects" (p.126). Yet there are only "few interactions that permit prescriptions" (Tobias, 1976, p.63) which is due partly to the shaky descriptions of treatment and to the neglect of the social dimension (see sections 3.1 and 3.3). But as well it is "the experimental strategy dominant in psychology since 1950" that "has only limited ability to detect interactions" (Cronbach, p.123). "The investigator delimits the range of situations considered in his research program by fixing many aspects of the conditions under which the subject is observed. The interactions of any fixed aspect are thereby concealed . . . . The concealed interaction may even wipe out a real main effect of the variable that chiefly interests the investigator" (p.124).

In the past few years, interaction has been interpreted as social interaction, and not only as a statistical interacting among variables. This has caused an important improvement of our insight into the teaching-learning process. At the same time, the isolated view of the main determinants has ceased.

Teacher-student relationships have been reviewed by Brophy and Good (1974) and Brophy and Evertson (1976). Much less work has been conducted on student-student relationships, though in small group work particularly the interaction patterns seem to exercise an influence on learning success.

2.2 Studies of real classroom situations

Many research outcomes drawn from the isolation of a laboratory context are not generalizable, and standardized tests are only of limited value for the interpretation of process: both of these factors have contributed to an increase in research studies of real classroom situations. Systematic observations of mathematics instruction, particularly long-term studies of learning processes in natural environments, have produced remarkable results. Soviet psychology of education especially has concentrated on teaching experiments, by which the learning of mathematics is studied under variations in the conditions of instruction.

Apart from the periodicals, the *Soviet Studies in the Psychology of Learning and Teaching Mathematics* (Kilpatrick et al., eds, 1969) is the best (English) source of information. It is the Academy of Pedagogical Sciences, created in 1943, that has a key function in educational research. In clear cut opposition to the Geneva School "Soviet psychologists ascribe a leading role to instruction. They assert that instruction broadens the potential of development, may accelerate it, and may exercise influence not only upon the sequence of the stages of development of the child's thought but even upon the very character of the stages. . . . changing conditions of instruction, and by varying these conditions, they demonstrate how the nature of the child's development changes in the progress. As a result, they are also investigating tests of giftedness and are using elaborate dynamic, rather than static, indices" (Kilpatrick, p. V). Thus the outcomes are presented more in terms of process descriptions than in psychometric measures, though the use of the latter seems to be increasing in the Soviet Union.

Slightly as yet, the opposite trend is becoming discernible in North America, where the number of case studies conducted under natural classroom situations has increased remarkably. A striking example is the work of Erlwanger, who employed a combined observation-interview method with six children for half a year, exploring their "ideas, beliefs and views about mathematics and learning mathematics" (Erlwanger, 1974, p.1).

2.3 Research interest on the teacher

The focus of research interest on the teacher – who is no longer regarded as an "intervening variable" in order to have a "pure" investigation of the effects of the treatment or of the student's learning – has increased the participation of teachers in research. In particular, the French IREMs (*Instituts de Recherche pour l'Enseignement des Mathématiques*), which have made a principle of the teachers' participation for reasons of innovation strategies, have accomplished pioneer work in this area. Similar work is being done at the IOWO (*Instituut voor Ontwikkeling van Wiskunde Onderwijs*) at Utrecht, Netherlands, and in many of the Teachers' Centres in England, as far as they have specialized in mathematics education.

Two remarks should be added to the emphasis placed here on the teacher's key role for the learning processes in the classroom and on the crucial role of his behavioural changes for the realization of innovation in schools.

The one is related to the necessity of finding out more about the teaching pattern of successful teachers and their interaction with students, not so much from the viewpoint of social psychology but more concerning the aspect of the transmitting and generating of mathematical meaning or cognitive schemata. This, particularly, since "the deliberate search for a good problem representation, or even a capability for generating and considering alternative representations does not seem to be a common part of the human problem-solving repertory" (Simon, 1975, p.18).

The other remark is related to the possible consequences of over-estimating the roles mentioned, and of the disappointment that the modesty of attainable changes in teaching might create. Since very recent research on change in teachers shows a high inflexibility of teaching behaviour and teaching strategies, one can foresee a new wave of content analyses and teacher-proof curricula. On the contrary, I think, it is the task of mathematics educators to contribute to an improvement of the teacher's decisions and of the student's learning (and vice versa: the teacher's learning and the student's decision!). The classification of educational research into "conclusion-oriented" investigations and "decision-oriented" investigations as proposed by the *Report of the Committee on Educational Research* (Cronbach and Suppes, 1969) less than ten years ago does not even include the decision-making of teachers.

2.4 Extension of the repertoire of research methods

One of the most conspicuous reactions to the meager relevance of educational research to practice and to the related problems of constructing theory has been an extension of the repertoire of research methods, which on the one hand announces a loosening of the hitherto rigid standards for the acceptability of methods, but on the other means that the integration of outcomes will become more difficult. The earlier non-acknowledgement of Piaget's informal research and the receding wave of formal follow-up studies of his statements support this. It is difficult to bring teachers to change – but it is the same with researchers. Therefore the main stream of research follows well-worn paradigms, tradition and routines, and only a small minority of researchers are engaging themselves in action research, descriptive studies or other informal or less formal approaches. Above all, this is true with dissertations, which form more than half of the research studies in mathematics education in the United States; these were 1486 in number in the period 1971-75 according to ERIC/SMEAC (Educational Resources Information Center / Science, Mathematics and Environmental Education Information Analysis Center).

Obviously the majority of contributors to research in mathematics education does this as an initiation rite only and continues work in other disciplines, if at all. Thus, in general, these studies are more rigid in methods, "collecting the most data in the shortest time", and are "of little value to developing educational theory" as critics state. More co-operative effort and planning "as part of broader general programs of research" (Shulman, 1970, p.392) would increase effectivity.

Another reasonable part of research has been conducted by larger institutes. Presumably there is a "critical mass" in terms of accumulated competence that facilitates interdisciplinary connections, provides for theoretical orientation, enables consistent long-term research programmes, and maintains animating discussions.

In the United States, where about 85 per cent of the research studies related to our theme were produced, the information-processing approach with its more open methods and simulation techniques is displacing learning theory and the struggle between cognitivists and behaviourists. This methodological broadening has caused increased attention to the descriptional systems, the "philosophies" that are fundamental to the different research approaches (see section 2.9).

Information processing approaches, though split into several colonies meanwhile, are clearly distinguished from behaviourist positions by "explicit attempts to describe *internal* processing" and from gestalt or cognitivist or Piagetian positions by their attempts "to translate 'restructuring' or 'logical operations' into temporally organized sequences of actions", i.e. to describe processes (see Resnick, 1975). The first coherent presentation was given by Miller, Galanter and Pribram (1960).

2.5 A need for theoretical orientation

Consequently, a strong need for theoretical orientation has emerged, whose power indicates a kind of crisis. The procedure of studying intervening influences from certain studies as main variables in subsequent studies is very common in the social sciences, but rather than integrating and concentrating, it is differentiating and ramifying our knowledge. Increasing understanding of the complexity of the learning process creates an increasing complexity of understanding, which is obviously difficult to deal with. For economic reasons alone, it is impossible to increase the number of variables in a study *ad libitum*, but it is equally impossible because of the interactions to keep all the other variables constant or just to neglect them. Additionally, in a rapidly changing modern society the validity of many research outcomes is limited in time (the historicity of statements). So substantial doubts have arisen as to whether general theories can be constructed along the ways favoured in recent research. Some researchers donot think it possible at all (e.g. Glaeser, 1975).

The number of articles in periodicals dealing with theoretical orientation has increased dramatically, mostly combined with criticism of methods.

> "The experimental psychologist, in seeking general principles applicable to 'persons-in-general' construes any form of interaction with persons to be a threat .... because it detracts from the purity of the principle." .... "The individual differences psychologist, on the other hand, seeks person characteristics that are stable over time and situation, the less environmental variability, the better. He, therefore, views any interaction of the person characteristic with the environment as situational variability, error, or otherwise detracting from the purity of the person characteristic." (D. Hunt, 1975, p.211).

> "Social scientists generally, and psychologists in particular, have modeled their work on physical science, aspiring to amass empirical generalizations, to restructure them into more general laws, and to weld scattered laws into coherent theory. That lofty aspiration is far from realization." (Cronbach, 1975, p.125).

The conclusion is inescapably, that we have to abandon "the myths of the general effects model and of personal constance" (Hunt) and that "the time has come to exorcize the null hypothesis" (Cronbach, p.124). Enduring systematic theories "are not likely to be achieved, systematic inquiry can realistically hope to make two contributions": " . . . to improve shortrun control" and " . . . to develop explanatory concepts" (Cronbach, 1975, p.126).

Thus the situation may be characterized by an intensive search for a frame of orientation – be it a theory or just a functional regulation of the research processes – a search in which many single researchers and groups at larger research institutes are engaged, e.g. at the *Shell Centres for Mathematical Education in Chelsea and Nottingham*, at the *Georgia Center for the Study of Learning and Teaching Mathematics* in Athens, and at the *Institut für Didaktik der Mathematik (IDM)* in Bielefeld (see also section 3.3).

2.6 The matter "meant", "taught" and "learned"

The learning of mathematics requires more than the availability of rote knowledge. This "more" can be described as meaning, as understanding (or insight), as adaptation to reality, etc. Research shows that this "more" is employed in the classroom in at least three forms: (a) as the structure of the mathematical discipline, *the matter meant*; (b) as the content of the teaching process shaped by the teacher's learned structure and routines, *the matter taught*; and (c) as the cognitive structure of the individual student, *the matter learned*. These three forms coincide in the ideal case only.

An important example for this tripartite reality is the study of Woods, Resnick and Groen (1975), who have investigated simple subtraction processes of second- and fourth-graders. Among five alternative models hypothesized for the performing, clearly one worked best, the "choice-model", which works by "incrementing up" or "decrementing down" depending on which has fewer steps. This model has never been taught, but it can be developed from the usual model taught, though it is more difficult due to the necessary choice. For German-speaking mathematics educators, the finding is of particular interest, since the choice-model works on the same principle as a teaching method of Herman Haase, called "auszählendes Rechnen" and published in 1898 for use in elementary school.

The distinction among mathematical structures, instructional structures, and cognitive structures seems to be useful for ongoing research as well as for teacher's diagnosis of students' learning. Lesh (1976) used the distinction for an interpretation of advanced organizers in their functioning for the "structural integration" of a concept.

What do these three forms, the matter *meant*, the matter *taught*, and the matter *learned*, mean for teaching and learning?

2.6.1 *The matter meant*

We have to abandon the image of the unchangeable subject mathematics which is passed on to the student by the teacher. On the contrary, the subject matter varies and changes in the course of the teaching process and in the individual learning process as well. Incidentally, this also supports the relevance of theories of social interaction for mathematics education (see sections 2.9 and 3.1).

2.6.2 *The matter taught*

This insight into the individuality of the structure and its development in the teacher and in the student is a strong argument against the search for *the* best treatment. Instead, the question may be put: Optimal for which students and under which accompanying conditions? Direct answers to this question are somewhat rare and are still in need of pragmatic interpretation.

Indeed, reviews of research also confirm, "that no general conclusion can be drawn regarding the relative efficiency of any one pattern for mathematics instruction" (Suydam in Bauersfeld, 1976, p.100). Reviewing thirty-six studies in which elementary school children in self-paced mathematics programmes were compared with traditionally taught children (Schoen, 1976, p.94) states: "Concern for the individual student's needs is very important indeed, but this approach to meeting those needs in the mathematics classroom is (or at the very least, has been to date) generally ineffective". The view of individual differences and self-pacing had been taken by few studies only, so there are no better answers, except the usual conclusion: "Further research is needed". But by the way, Schoen's reasons for predicting "a natural death" to self-paced instruction deserve more common consideration, since they apply as well to other instructional approaches: "1. Cost and the fact that even the best programs do not improve achievement; 2. Heavy demand on the teacher's time and nervous system; 3. Excessive amount of test-taking time needed for student evaluation leaving very little time for learning; and 4. Isolation of naturally social children." (Schoen, p.92). Similar summaries and comments are concerned with discovery as opposed to expository methods, laboratory approaches, etc.

"The teacher who learns as many instructional modes as possible, identifies and diagnoses pupils' needs and abilities, and uses this knowledge to individualize instruction may very well get the best results" (Suydam and Weaver, 1975, p.2-6). But to be able to do this identifying, diagnosing and using, the teacher is in serious need of support from research and development.

2.6.3 *The matter learned*

Furthermore, the students develop interpretations and strategies on their own, which may remain hidden for a long time, although active, if not detected by means of a mistake or an unusual solution. The extent to which the teacher can adapt to such situations characterizes the quality of teaching. For improvement, a long-term diagnosis and assessment of the student's development is required, particularly since the time required for learning increases with the complexity of the subject matter structures and the rules for their generation.

As an example, pupils' individual ideas of what geometrical transformations mean can differ heavily depending on the preceding teaching-learning processes. Error analysis of criterion-referenced tests (Bauersfeld, 1975) reinforced the assumption that only a minority of the pupils employed adequate conceptions of a 180° rotation in the plane, while the others performed consistently corresponding to ideas of mirroring or folding down (three-dimensional rotation). The teachers were not aware of these differences in understanding although the instructional structures, examples and models used in the classroom may have supported the creation of the differences. This led to the idea of changing the function of the criterion-referenced tests by restructuring the items to work for the diagnostic needs of the teacher (and the pupil).

2.7 The embodiment and the matter learned

Because of difficulties with the formulation and hierarchization of objectives and because of difficulties with the validity of related tests, doubts have arisen as to whether the different models for teaching content – this means the modes of representing and embodying the "matter meant" – really transmit the identical content. There is good reason for the assumption that different models can bring the student to the construction of concepts that differ in range and content and appear identical only because of uniform symbolical description and of normed technical language. This is shown clearly through poor generalizations by the student, and through difficulties in solving exercises involving applications. Since the lower a student's achievement, the more he depends on the material and its connotative background, disadvantages are caused precisely for those students for whom embodiments are made. Particularly in pre-school and primary education this observation makes necessary a revision of so-called introductions or fundamental experiences and suggests the development of compensating aims for a more complete understanding. It also makes necessary a critique of achievement testing.

That individually different effects in concept formation are caused by different embodiments or presentations of the same mathematical structure is in accordance with other findings. There are internal influences from personal characteristics or styles related to individual differences in perceiving, encoding and decoding of information, storing and retrieving of information, as well as in affective or social dimensions. Likewise there are external influences, such as from the processing of the embodiment through the teacher, his instructional routines, as well as from the relationship to the classmates and from other environmental factors. Hitherto researchers have paid more attention to the effects of general classifications, e.g. "subtractive vs. generative approach" (Walbesser et al., 1976), "figurative vs. operative concepts" (Piaget), to the role of "negative instances" (Shumway, 1974), and so on, *across* differences in content, situation, and person characteristics. Correspondingly, we need more information the other way round.

2.8 Implications of different views of mathematics

One cannot expect sound statements about the learning process without a proper elucidation of what has to be learned. Many empirical investigations give the impression that mathematics is an enduring body of knowledge, a hierarchically ordered set of fixed statements and rules about concepts which are related to one another like knots in network. Mathematicians and mathematics educators probably will not agree with this narrow picture of mathematics as a theory of static structures; instead, they may appreciate the more dynamic metaphor of "mathematics as process" (Freudenthal) or "mathematics as a language" (Papert). No doubt such fundamentally different images of the "matter meant" will influence the "matter taught", this means that they will produce different forms of mathematics teaching. The problem gets more complicated if one takes into account the different fundamental philosophies that mathematicians have about their own subject matter (e.g., intuitionists versus rationalists), which stand in opposition to popular ideas about mathematics. In the empirical sciences there is still not much reflection about these differences. Some evidence is to be found in more informal studies, from which it is clear that different views of mathematics generate a different comprehension of e.g. (mathematical) "generalization" or "abstration", or what "comprehension of a concept" may mean, or what is meant by a "sufficient solution" to a mathematical problem.

Striking evidence can be drawn from Freudenthal's *Mathematics as an Educational Task* (1973) with his key question about the "constitution of mathematical objects", from Papert's work related to pupils' "communication" with computers (1975), from Fischbein's work on the development of probabilistic thinking (1975) and his key question about the "acquisition of intuitions".

2.9 Implications of different views of the teaching-learning process

On the other hand it is impossible to conceive of research related to the learning of mathematics that is not shaped by some fundamental view of the learning process, or better, of the teaching-learning process. This is, at least for the learning of mathematics, not just a question of different descriptional systems which are applied to the one identical subject matter. Since at school mathematics is a product of communication (a variation of a statement by Watzlawik on "reality" in general), the product is deeply related to the structure of this social or communication situation. Interest in clarifying this dependency has increased very recently. Easley (1976) has presented a sophisticated study of seven modelling perspectives on teaching and learning, and Joyce (1976) has described four different families of model: "social interaction models, information-processing models, personal models, and behaviour modification models". Since the models differ in their view of man, the nature of his knowledge, and his learning, they indeed define different issues of research and of education. Incidentally, this is an important and essential reason why it may be impossible to integrate different research outcomes on the "same" theme.

2.10 Increasing interest in conditions for students' learning

Though the process of mathematics learning cannot be described adequately from the point of view of the learner only, there is an increasing interest in the conditions for the success of students' learning, particularly in the structure of mathematical abilities and the rules of their development. This theme comprises the relatively largest number of research studies, among which are hundreds of follow-up studies to Piagetian experiments (mainly concerned with conservation). Attempts to split up mathematics ability, which in tests is often treated as a unit, differ in their outcomes according to the approach used: a re-analysis of factor-analytical studies produces models of abilities similar to well-known models for the structure of intelligence (Roeder, 1974, and Treumann, 1974); case studies using fixed sets of tasks with high achieving students lead to models of abilities which are described by generalized mathematical processes (Krutetskii, 1976). Hitherto the results are not detailed enough to provide a more reliable diagnosis for ability grouping in mathematics education, but they do assist the improvement of individualization.

On the other hand, a large variety of variables influencing mathematics learning has been investigated, such as the previous learning experiences of the student, his self-esteem and self-concept, his styles of perception and information processing, the reinforcing behaviour of the teacher, and so on, to mention a few of the most important. Several of these variables turn out to be very stable during school-life, but surprisingly, others are rather unstable, which means they are dependent on the learning situation and on the interacting persons. The results of this research support scepticism concerning the usual practice of streaming according to achievement measures, since such streaming might cause a fixation of unstable characteristics. Furthermore, important consequences must be drawn from this research for pre-service teacher training (see Chapter VI).

3 CRITIQUE AND CONCLUSIONS

The critical assessment of developments during the last few years and orientations for future work deduced from it are grouped into the following three dimensions, which characterize aspects rather than strictly separate issues.

3.1 The social dimensions of the mathematical learning process

Here three aspects are to be stressed, which need thorough analysis and further investigation.

3.1.1 *Learning through dialogue*

If it is true that mathematics can be learned only through dialogue, then the rules for communication situations, and more generally for social relationships between persons, are valid for these processes also, and these presumably apply to issues specific to the subject matter. From this point of view, the learning of mathematics as the learning of meaning requires the "negotiating" of meaning for the student. In order to prevent possible misunderstanding: the point is not that students share the decision as to what is mathematically "true", but it is that only in communication with others can the student test the appropriateness of his constructs and correct "his" concept. To a large extent social situations are determined by the preliminary experiences, the value systems, and the expectations of the persons participating in them. Therefore, rather than perceiving objectively what a student does, the teacher sees it through the pattern of his expectancies and through the attributions of causality he makes. The same holds for the student's relationship to the teacher. Careful further investigation is needed to derive implications for the individual development of the student, for the assessment of his achievement, for ability grouping, and so on. That "specificity to situations" is used increasingly as an explanation for contradictory findings or non-significant results in research studies confirms the importance of this point of view.

3.1.2 *Inference by the general cultural context*

Without doubt the cultural bias of tests, which has long been discussed, is also involved in the accessibility of mathematical concepts. Numerous accounts of experiences with the introduction of "new mathematics", particularly in developing countries, show this. Positive and negative interference in comprehension is caused not only by differences among languages in grammar or semantics but also by the general context of different cultures. "Cultural differences in thinking indicate differing uses of general cognitive skills in specific situations" (Report of the Unesco-Unicef–SEPA seminar, Nairobi, 1974, p.48). Such insights have raised the sensitivity of researchers in developed countries to the possible interference with learning caused by different social backgrounds. This refers to the whole range of possible differences: from general patterns of behaviour to the connotative meaning of concepts. Thus, a sociology of knowledge and of the relevant strategies of behaviour in mathematics education has to be developed.

3.1.3 *Identification of individual differences*

The problems of a proper mathematical development and education of the individual student will not be adequately solved until careful analyses of the causes of individual differences are made, diagnostic aids for the teacher to identify differences are developed, corresponding preparation during the teacher's training is provided, and conclusions for school organization are drawn.

3.2 The content dimension of the mathematical learning process

Here one must point out the deficits indicated in sections 2.6–8. It seems necessary to specialize the investigation of learning processes. This means that we have to care about how certain concepts are being learned – instead of generating general theories of concept formation. We have to find out about the function certain concepts have for certain students in sufficiently determined situations, and about the relations and meaning that are developed from it in a long-term view. New ideas are required for the description of the different forms or developmental stages of a concept, particularly for the description of the "matter taught" and the "matter learned". The same is true for the methods of investigation, which should not be restricted to a replication of available routines. In this connection it might be important as well to investigate the intuitive patterns of teaching and rules of interaction that successful teachers employ and generally do not reflect upon.

3.3 The theoretical dimension of research

The precariousness of the situation in the empirical social sciences seems to be near to a crisis and has led to the assumption that, in Thomas Kuhn's terminology, a change of paradigm is going to occur. This is especially true for psychology. Research methods that mainly follow the paradigm of the natural sciences are being questioned. But the difficulties are probably not so much with the methods employed as with defining the area for their adequate application. The outcomes of an investigation are always determined in advance by the fundamental theories on which the hypotheses have been developed. The weaker the theoretical basis of empirical work, the more severe the difficulties it encounters and the less it can lead to new theories. If a change of paradigm is on the way – and much suggests it is – then it will be a fundamental change in the understanding of the object and aim of research: The behaviouristic paradigm has run into difficulties due to its definition of man as driven by unknown forces, as reacting predominantly to external stimuli, and as an object whose behaviour can be determined by control of his environment. This view, for example, admits self-reflection as a hypothetical construct only, as a mentalistic intervening variable. The opposing view is of man as a reflecting subject, using his subjective theories for the explanation of his environment and deducing his activity from these explanatory theories. In contrast to the behaviouristic models, this model is applicable to itself: the non-symmetrical subject-subject relationship is abandoned in favour of a principally symmetrical subject-subject relationship. The subjective individual explanatory theories can be criticized in terms of psychological rationality (e.g. theory in scientific terms), and vice versa. That this symmetry formed a key characteristic in the European tradition of hermeneutic psychology might be stated here with some satisfaction.

Consequences can easily be drawn but will take time to be realized: the construction of theories related to the teaching-learning process has to include the knowledge of all of the participants – teacher and students – about themselves. Theory therefore is possible only as (meta-) theory about subjective explanatory theories. This can be understood as a main orientation for future research work.

4 PROSPECT

A review of the more than 3000 research studies relevant to the theme leads one to the conviction that mathematics educators – being transmitters between research and school practice – cannot restrict themselves to selecting from the rich body of knowledge produced by the relevant sciences and to interpreting these outcomes for practical use. On the contrary, by defining their own questions and problems considerable inadequacies appear; a shortage in the midst of abundance. Furthermore, since relevant research in other disciplines is conducted under aims and interests that are not directly aimed at the promotion of mathematics learning, and since it is difficult to get a discipline to adapt its research strategies for use in other disciplines, mathematics educators will have to develop their own approaches. Because of their transmitting function, they will not be able to restrict themselves explicitly to the stringent methods of sociometrics, nor will they be allowed to get lost in the lack of control and non-replicability of pragmatic action.

Research and development within a discipline of mathematics education therefore will have to develop its own standards and methods. This means that an adequate precision will have to be developed which on the one hand makes the outcomes accessible to the practitioner without further interpretation and on the other hand is open to scientific control.

But the overestimation of mathematics education as an omnipotent trouble-shooter has to be avoided. Mathematics educators and teachers have to learn:

(a) how to cope with a highly complex reality, i.e. with a variety of explanatory theories that are powerful only in certain respects, but that are not integratable or that are even contradictory;

(b) how to work with a variety of general descriptive systems (ideologies or philosophies) that allow optimal description only in certain respects in spite of their aspiration to totality, but that do not work well in other respects; and

(c) how to deal with a variety of methods for research and for teaching that rarely match, but that are as subject to history as the descriptive systems and the explanatory theories.

It is legitimate for researchers to embed investigations in one and only one system (philosophy, theory, method) because scientific truth is valid only in such a system, and it is an unavoidable necessity for the teacher to make one-sided decisions in a given context and under the severe pressure of time; but neither of these approaches may be permitted the mathematics educator in his transmitting function. He should not reduce complexity by following current modes of research or public opinion, nor should he become lost in a poorly oriented pragmatism. He has to develop his own self-concept.

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PANEL

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Chapter XI

A critical analysis of the use of educational technology in mathematics teaching

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INTRODUCTION

This chapter has three sections: 1. A general view of technology and its import for mathematical education; 2. A critical analysis of the individual technologies, their pedagogical attributes and trends in their uses; and 3. The past, present and future of the use of technology in mathematics teaching.

The purpose of section 1 is to set the stage for the chapter. Specifically, the special characteristics and attributes of technology in general will be considered – with special reference being given to the types of educational and pedagogical problems that technology should be able to help solve.

The purpose of section 2 is to (1) delineate the pedagogical attributes and limitations of the individual technologies, (2) provide information about important efforts to take advantage of these technologies for the purpose of improving some aspect of mathematical education, and (3) try to uncover significant trends regarding the role(s) of these technologies in the classrooms of the future. For the convenience of the reader, the list of contents of section 2 is stated below on page 216 at the beginning of the section.

The purpose of section 3 is to provide a comprehensive and rational analysis of trends in the use of educational technology in mathematics classrooms. This analysis will take into account multi-media approaches, the types and levels of mathematics for which the use of technology seems most appropriate, teacher/student attitudes toward the use of technology and other relevant considerations.

1 A GENERAL VIEW OF TECHNOLOGY AND ITS IMPORT FOR MATHEMATICAL EDUCATION

The interpretation of educational technology as it is employed in this report will refer to " . . . the media born of the communications revolution which can be used for instructional purposes alongside the teacher, textbook, and blackboard" (Tickton, 1970). The rationale for this approach is that the media of interest, including films, filmstrips and slides, television, and computers have entered education independently, and still operate more in isolation than in combination.

The *book* was the major technological advance of five centuries ago, and just as it had a revolutionary effect on education as it had existed up to that time, many educators feel that recent technological developments, particularly those involving computers, have the potential for equal or greater impact on the educational enterprise. The cause for such optimism, of course, is a direct outgrowth of the pedagogical attributes of the newer technologies, a topic that has been discussed extensively in the literature of the field by Green (1972) and others. The major attributes may be summarized as follows:

1) Media can do things – can create learning situations – that cannot otherwise be accomplished (like bringing current events into the classroom).
2) Media can be used to present information in a variety of ways, which best meet particular learning objectives.
3) By varying the medium, information can be presented to groups of many differing sizes – from individuals to national audiences – and it can be presented simultaneously to those differing audiences.
4) Media can make learning more effective by increasing the realism, the dynamics, and the attitudinal impact of information; it can increase the motivation to learn.
5) Some media, such as television, can make the best teachers and learning situations available to more students than would otherwise be possible.
6) Media can extend the limits of the learning situation by reinforcing and expanding the experience and background of teacher and student. In addition, they can extend boundaries imposed by school plant and geographical location.
7) Media can allow for individualization of curricula by presenting the same information, or variations, to different students at different times.
8) Media can allow students to work under many situations without teacher guidance or supervision, freeing the teacher for individual assistance.
9) Media can, in some cases, be employed to direct information to some people in shorter-than-conventional periods of time.
10) Some educational objectives can be realized more economically by using media rather than by conventional means.
11) The use of media forces educators to examine their goals and objectives more closely than before.

No single medium enjoys all of the attributes, however. Each individual medium (technology) has its own special pedagogical capabilities and limitations, and so the educator must be aware of them in order to effectively plan for their use, either singly or in association. Section 2 deals in part with this matter in addition to exploring the nature of current developmental efforts and trends in the uses of the newer technologies.

2 A CRITICAL ANALYSIS OF THE INDIVIDUAL TECHNOLOGIES, THEIR PEDAGOGICAL ATTRIBUTES AND TRENDS IN THEIR USES

2.1 The non-computer-related instructional technologies

2.1.1 *Films*
  (a) Pedagogical characteristics and considerations
  (b) Some prominent developmental efforts
  (c) Analysis/trends

2.1.2 *Filmstrips/slides/cassette tapes/video recordings*
  (a) Pedagogical characteristics and considerations
  (b) General survey of developmental efforts
  (c) Analysis/trends

2.1.3 *Television/Radiovision*
  (a) Pedagogical characteristics and considerations
  (b) Survey of significant developmental efforts and uses
  (c) Analysis/trends

2.2 The computer-related instructional technologies

2.2.1 *Computer-managed instruction*
  (a) Pedagogical characteristics and considerations
  (b) Survey of significant developmental efforts
  (c) Analysis/trends

2.2.2 *Computer-assisted instruction*
  (a) Pedagogical characteristics and considerations
  (b) Survey of major developmental efforts
  (c) Analysis/trends

2.1 The non-computer-related instructional technologies

2.1.1. Films

(a) *Pedagogical characteristics and considerations*

Proponents of the film medium give many reasons for the use of motion pictures in the classroom, and many of these reasons are as compelling for mathematics teaching as for any other subject.

As every experienced teacher knows, the problem of *motivation* is critical in the teaching of mathematics. There is no question that well-designed films can do much to arouse student interest and create a level of classroom excitement that is difficult to achieve in any other way. Good examples of films which must be rated high in terms of their motivational qualities are the thirty student films in the series entitled "Elementary Mathematics for Students and Teachers" which was produced by the National Council of Teachers of Mathematics in co-operation with the General Learning Corporation, both of the United States (1).

The motion picture also affords an opportunity to provide students with experiences which they could not otherwise enjoy. Rare, and occasionally unique, phenomena that are virtually impossible to provide in classrooms in any other way can be made readily available on film. The film "Symmetries of the Cube" (2), for example, makes possible the study of the cube and regular octahedron and their symmetry group through reflections in an octahedral kaleidoscope. Precision kaleidoscopes are rare, extremely expensive, and most difficult to build. Thus, this film brings into classrooms that which is truly impossible for teachers to provide. The computer-generated film "Space-Filling Curves" (2) does likewise in exhibiting the picture of the step-by-step construction of two remarkable examples of curves that pass through every point of a square.

The motion picture offers special pedagogical advantages that arise from its most obvious characteristic, namely that of creating the *illusion of motion.* Dynamic phenomena and related geometry are therefore prime topics for study via films. The techniques of *animation* and *stop motion* can be utilized for various types of portrayals. A variety of interesting examples of such utilization are contained in: "The Seven Bridges of Konigsberg" (2), which uses animation to show how a mathematical model, namely a graph, is obtained as an abstraction of a concrete problem; "Four Line Conics" (18), which employs animation to generate a one-parameter family of conics, thus showing "infinitely many" curves in a matter of seconds; "Possibly So, Pythagoras" (2), which demonstrates transformations of regions by means of animation; and "Topology" (3), which exhibits a spherical lump of clay changing its shape, first to a vase, then to a pancake, a worm, a mushroom and, finally to a cube, thus providing the viewer with a lucid example of topological equivalence. Finally, it should be observed that the availability of motion allows for the possibility of exhibiting and employing the role of continuity in geometry. Examples of this appear in "Possibly So, Pythagoras" and in "Orthogonal Projections" (3). In the former, animation is used to exhibit a triangle transforming so that one of its angles increases continuously from 0 to π ; the viewer is then firmly convinced that, at some point, the figure must be a right triangle. In the latter film, an elliptical region is transformed continuously so that its (original) major axis shrinks to zero while its minor axis remains constant. By virtue of this scene, the viewer realizes that the region must have been circular at some stage of the transformation.

More recently, animated films have been produced by photographing a computer generated video display. This technique would appear to lend itself particularly well to the production of films depicting special mathematical functions and their characteristics. Other special aspects of motion picture technology which have pedagogical import are time lapse photography and microcinephotography. As an example of the latter, children will be fascinated by the close-up shots of bees manipulating wax in "Mathematics of The Honeycomb" (4), a film which shows that by using hexagonal prisms, bees minimize the amount of wax needed to build a honeycomb.

Though films have many pedagogical advantages, they also have limitations. Generally speaking, films do not provide *responsive* or *adaptive* learning environments; that is, they are not built to accept, or respond to, student inputs nor are they designed so as to be capable of making "real time" adjustments based on the reactions of the learners who are viewing them. It follows, therefore, that the typical classroom film should not be regarded as a lesson in itself. Both preparatory and follow-up activities are important. Students should be given information about what to expect in a film and, after seeing it, should have an opportunity to discuss its content. In other words, the experienced teacher will make use of a film as a spring-board to further learning.

The proper incorporation of films into classroom settings, therefore, necessitates considerable planning and preparation on the part of the teacher. So, too, do the problems of setting-up and operating conventional film projection equipment, and collectively these matters have served as somewhat of a deterrent to the wider use of films for the purpose of teaching. Fortunately, the development of the 8mm cartridge projectors during the 1960s has helped ameliorate the problem. Loading a cartridge projector is simply a matter of pushing a cartridge containing a film into a slot. The film, then, is ready for immediate showing. As it is shown, it is automatically rewound in the cartridge, and so the problems of threading, adjusting, and rewinding are nonexistent. Cartridges range in capacity from four to thirty minutes of film. Some projectors are of the normal front screen type while other make use of rear screen projection. An advantage of the latter type is that little darkening of the room is required for showing. Many cartridge films are silent, but both optical and magnetic sound are available.

The majority of cartridge films are short, single concept "loops". These are ideal for individual or small group instruction. The simplicity of operation of the cartridge projector means that almost any child of school age can operate one. Moreover, since the film rewinds itself in the cartridge, individual students can view it as many times as they wish. It also should be pointed out that the cartridges can be stored on shelves or in boxes, and thus can be as accessible to students as library books or other portable learning resources. It is also now possible to transfer regular 16mm films to video cassettes which have the same ease of operation as cartridge films. The future development of videodiscs may afford similar possibilities.

(b) *Some prominent developmental efforts*

Most of the significant mathematical films that are now available were developed during the past twenty years, and many of these have been produced by, or in co-operation with, professional mathematics organizations or leading curriculum development groups.

In the United States, an ambitious film project was undertaken in the late 1960s by the National Council of Teachers of Mathematics (NCTM) in co-operation with General Learning Corporation. The mathematical content of the films, designed mainly for students in grades 3-8, was determined by an NCTM advisory panel, but the film producer was given a great deal of freedom in deciding how to present basic ideas. The result was a series of thirty fascinating films in which the mathematical concepts are discussed by cartoon characters. The films vary in length from about six to thirteen minutes. Accompanying the student films are twelve films designed to be used for teacher education. In these latter films, important mathematical concepts are discussed and they also include demonstrations of teaching children in classroom situations.

During the 1960s, the Mathematical Association of America (MAA) also was active in producing films to support mathematics instruction. As one of their projects, they produced a series of fourteen animated films designed to be used in a beginning calculus course. Some of the titles in this series are "A Function is a Mapping", "Limit", "Area Under a Curve", and "The Theorem of The Mean". The MAA also filmed a series of lectures by prominent mathematicians. Examples of films in this series are "Let Us Teach Guessing" in which Professor George Polya leads an undergraduate class to discover the number of parts into which 3-space is divided by five arbitrary planes; "Challenging Conjectures" in which Professor R. H. Bing describes the history and background of five well-known problems; "Measure and Set Theory" in which Professor Stanislaw Ulam defines the concept of measure and considers related matters of existence and uniqueness; and "Matching Theory and The Marriage Theorem" featuring Gian-Carlo Rota (2).

(c) *Analysis/trends*

Motion pictures designed for classroom use have been around for more than half a century, but at this point in time it cannot be asserted that they have met with a great deal of acceptance on the part of mathematics teachers. Indeed, practising mathematics teachers who have never used a film in the classroom are probably in the majority. Yet, the number of mathematical films that are available seems to be increasing constantly. For example, in the United States, the National Information Center for Educational Media (NICEM), established at the University of Southern

California in 1966, has a computerized data bank from which it produces reference volumes of educational media. In its first *Index to 16mm Educational Films*, there were approximately 500 titles listed under the mathematics subject heading; in the Fifth Edition of the Index, however, there were more than three times as many mathematics listings. Moreover, the latest NICEM *Index to 8mm Motion Cartridges*, published in 1974, contains over 1000 mathematics titles. A closer examination of the foregoing data reveals the fact that the commercial sector of the educational enterprise is taking a greater interest in the film market which, in turn, suggests that the classroom use of films is on the rise, though probably not significantly so. Another possible trend is toward the development and use of film recordings of computer-generated video displays. A project along these lines has been conducted at Carlton College (5) entitled "Computer Graphics for Learning Mathematics". Included in the products of this project are a collection of thirty-three super-8 films.

In terminating this aspect of the report, it should be noted that the problem of quality in mathematics films has been a persistent one, a fact that surely has been an impediment to their use. In spite of this circumstance, however, the general trend has been toward the production of higher quality films; this outcome is undoubtedly the result of the participation of a greater number of mathematicians and mathematics educators in planning and overseeing film construction.

2.1.2. Filmstrips/slides/cassette tapes/video recordings

(a) *Pedagogical characteristics and considerations*

A filmstrip is a collection of still pictures on a strip of 35mm film. Usually there are between twenty-five and sixty frames on a strip. Filmstrips are relatively inexpensive and because of their size they require little storage space. Filmstrip projection equipment also is low in cost, easy to transport and operate, and generally durable.

Mathematics teachers seem to make more use of filmstrips than motion pictures, a circumstance that can be accounted for in part by the fact that filmstrips are cheaper and easier to store, have access to, and use than films. Another possible reason for this state of affairs is that many topics in mathematics must, by their very nature, be developed in a particular sequence. When a filmstrip is being shown, each frame can be viewed or discussed as long as necessary, the teacher can stop and answer questions as they arise, and the strip can be backed up, if desired, to examine previous frames. Another advantage of filmstrips is that in dealing with topics involving diagrams much work on the chalkboard can be saved, often with a significant improvement in quality.

Filmstrips are flexible in that they can be used with an entire class, or they can be used on an individual basis. In the latter regard, some filmstrips have been designed to elicit responses from the student and must be used with a special type of viewer. The procedure calls for having the student respond to multiple choice questions that appear on the viewing screen by pressing appropriate buttons on the projection device. A buzzer sounds if the choice is correct.

Some filmstrips also are co-ordinated with sound recordings, a combination which provides for an even richer learning environment. Individual or groups of students can employ such instructional packages in much the same way as the cartridge films described earlier. The sound recordings that accompany filmstrips are often of the cassette type, a reel-to-reel audio tape encased in a plastic container measuring about 6 cm by 10 cm. Cassette tapes of the foregoing type are now often marketed separately, but almost always are co-ordinated with some other type of learning materials such as study booklets. Typically students listen to the tapes and respond in a paper and pencil mode.

Essentially, most of what has been said about filmstrips applies to the 35mm slide technology. Slide projectors are relatively inexpensive, though they probably are prone to require more care and maintenance than filmstrip projectors. While the actual storing and handling of slides is a bit more cumbersome than filmstrips, the slide technology does have the special virtue of permitting changes and additions to teaching sequences on an *ad hoc* basis. Some teachers also may be able to produce their own 35mm slides of satisfactory quality.

(b) *General survey of developmental efforts*

Apparently the concerted efforts at film-making by recognized curriculum groups have not been duplicated in the production of filmstrips, slides, audio-tapes, or video-recordings, either singly or in combination. This is certainly the case in the United States. Among the noteworthy efforts which have been undertaken, however, are three university-level projects now under way in the United Kingdom:

(1) *The Development of Tape-Booklet Sequence for Self-Help Remedial Instruction in Mathematics for First Year Undergraduates at Edinburgh University.*

This work is intended as a supplement to lecture material and not as a replacement (6).

(2) *The Development of Filmstrip, Cassette-Tape Materials at The Department of Mathematics, University of Glasgow.*

This work is designed to supplement CAL (Computer-Assisted Learning) materials being prepared for first year students of mathematics at the University of Glasgow (7).

(3) *Video-Recorded Lecture Courses at The Department of Mathematics, University of Strathclyde, Glasgow.*

This project is based on video-recordings of carefully prepared lectures given in a studio without a live audience. Several complete courses have been prepared and in particular the course for the large general first year class in mathematics. Each lecture session consists of a first part of film lasting twenty minutes, a break of ten minutes during which questions can be addressed to a member of the staff who is present in the lecture room, and a second part of film, also lasting twenty minutes. The recordings are backed up by complete notes, problem sheets, tutorials, exercises and examinations. The project was initiated at a time when it was difficult to recruit competent lecturers, but has been successful enough to be continued into the present period of serious job shortage (8).

(c) *Analysis/trends*

The Fifth Edition (1975) of the NICEM *Index to 35mm Educational Filmstrips* lists over 3000 titles in mathematics, covering virtually every topic from beginning number concepts to calculus. Most of these filmstrips have been commercially produced and typically are designed to accompany learning materials of another type, such as textbooks.

As indicated by the nature of the projects described above, there is a distinct trend toward the development of *multi-media* approaches, even though the movement has not yet had a significant impact on the teaching of mathematics. The relatively limited interest that has been manifested up to this point in time may largely be attributable to deficiencies in teacher education programmes in regard to the use of technology. In any event, multi-media learning packages have a lot of attributes and natural attractiveness, and the long term prognosis would seem to call for their increased development and use.

2.1.3 Television/radiovision

(a) *Pedagogical characteristics and considerations*

Fundamentally, television enjoys the same basic pedagogical characteristics as motion picture films. Accordingly, as in the case of the use of films, the incorporation of television into the classroom requires special planning and preparation by the teacher.

The conditions and pedagogical considerations undergirding the use of broadcast television in the classroom are somewhat different from those concerning the use of films, however. As Bending (1970) has stated: "Film is a medium that is used by the teacher and is under his control. Television, at a National level, is essentially a medium that uses the teacher". The point is that the teacher, or school, that wishes to make use of a television programme must arrange class schedules to conform to broadcast times, and generally must build other aspects of the instructional programme around the broadcasts; thus, television imposes a more rigid structure on the classroom and the teacher. One major consequence of this fact is that there is a need for considerable classroom materials in support of television broadcasts. Along these lines, it was noted in the *Proceedings of the 3rd European Broadcasting Union (EBU) International Conference on Educational Radio and Television* (1967) that associated teacher material should provide the philosophy and aims of series and individual shows, lists of materials teachers should have on hand, statements of any before-show preparation that should be given the class, lists of unfamiliar vocabulary, answers to any questions that are posed, indications of possible follow-up activities, and so on. Furthermore, it was pointed out that student materials should include visuals for review and discussion, lists of books for further reading, course books for later class-work, visuals students can take home, programmed self-correcting materials when no teacher is available or the teacher is not qualified for follow-up, activity material for games or experiments, and question papers.

It also should be recognized that due to the potentially wide impact that a television programme can have on the educational enterprise, it is imperative that investments be made to assure high quality programme development from a curricular perspective. To do otherwise is to risk spreading mediocrity to an extent that would otherwise be impossible.

On the surface, the importance of the visual dimensions of learning and using mathematics would suggest that radio technology would have a very minor impact on mathematics education; but the concept of *radiovision*, radio accompanied by slides and printed materials for classroom use, is on the rise, and promises to be an effective means to reach students in far-flung areas who are out of reach of other broadcast or basic communications media. Radiovision requires basically the same types of auxiliary support materials as those described for television.

(b) *Survey of significant development efforts and uses*

There is a multitude of important television and radiovision projects under way in mathematics education, representing various countries throughout the world. In the *Proceedings of the 3rd EBU International Conference on Educational Radio and Television* (1967) a list is given of radio and television programmes in mathematics that were reviewed at the Conference. The list includes programmes from Denmark, Australia, France, Hungary, Ireland, and the United Kingdom.

The British Broadcasting Corporation (BBC) has been particularly active in producing mathematics programmes. David Roseveare (1974) has reported that:

"Since 1957 there have been nearly 300 mathematics programmes for schools in 11 different series on BBC television. This does not include repeats (which would put the figure at well over 1000) nor any of the Further Education series, let alone the Open University or ITV."

Among the mathematics programmes produced in recent years in Britain by BBC are: "Maths Workshop" for 9-11 year olds; "Mathshow" for 11 year olds; "Statistics in Perspective" and "Secondary Mathematics Individualized Learning Experiment" (SMILE).

Other countries that have recently mounted important television or radiovision projects in mathematics include the Federal Republic of Germany where *Südwest-Funk*, Baden-Baden in co-operation with the *Institut für Mediendidaktik, Erziehungswissenschaftliche Hochschule,* Rheinland-Pfalz (9) has developed television programmes for both fifth and seventh graders, and now has a programme for eighth graders under development; Brazil, where a multi-media system for teaching mathematics at the junior high school level has been developed which incorporates television as one of its basic components (17); Japan, which through the government-sponsored *Nippon Hoso Kyokai* (NHK) has developed and broadcasts elementary, secondary and senior high school television programmes in mathematics, and also secondary and senior high school radio programmes; the United States where the Educational Development Center (10) has developed an experimental mathematics programme (Project One) consisting of sixty-five half-hour television programmes designed for the primary grades which was aired during autumn, 1976, from six major United States cities, and also where the University of Maryland has prepared thirty half-hour television tapes intended to serve as an aid in teaching an introductory course in the uses of mathematics in the biological, social and management sciences; and Hong Kong which Marjorie J. McGilvrey (1976) claims has ". . . the best [television] organization and the soundest educational base of any system I saw around the world . . . reaching 335,000 elementary school students a week".

(c) *Analysis/trends*

In the report of the 3rd International Conference on Educational Radio and Television, written in 1967, it was stated that both radiovision and television services were on the increase, as were the number of programmes available and the number of teachers willing to use them. There now appears to be little doubt about the accuracy of this forecast. Though much of the activity has occurred in the so-called "developed countries", there would seem to be cause to believe that both radiovision and television, and perhaps in that order, could play important roles in the improvement of educational systems in the developing parts of the world where rapid educational progress is sought.

2.2 The computer-related instructional technologies

2.2.1. Computer-managed instruction

(a) *Pedagogical characteristics and considerations*

One of the most rapidly growing uses of the computer in education today is in the realm of what is widely becoming known as "computer-managed instruction" (CMI). As the term suggests, this type of computer support utilizes the computer as a manager; student achievement assessment, management of classroom learning, student records, and information retrieval typically are considered to fall within the sphere of CMI.

As has been pointed out by Allen (1975), CMI is perhaps best conceptualized as constituting a class or category of functions. A summary of the primary functions as identified by Allen and others is given below.

Standard features:

1) *Data storage*: student information such as student name and identification number, test scores, test dates, and so on.
2) *Report generation:* based on data saved; group and individual achievement reports, and so on.
3) *Detailed performance assessment*: for evaluation of instructional materials; detailed assessment of student learning helps identify weak instructional segments, thus assisting in the evolution of an improved instructional program.
4) *Test generation*: construction of tests that are geared to specific learning objectives and conditions of previous student achievement.
5) *Response judging/scoring*: of tests that are administered.
6) *Results feedback*: of student performance on tests; often detailed information is provided indicating those learning objectives which have been mastered and those which have not.
7) *Record keeping/data retrieval*: primarily for teacher use; data relative to various aspects of the instructional system to be used for purposes of evaluation.

Advanced features:

8) *Tailored testing*: highly sophisticated procedures for adjusting testing strategies in "real time", depending upon student performance; a tactic for more efficient testing.
9) *Study prescription generation*: the capability to produce a list of learning resources that students should consult to help them learn a specified set of ideas; the prescriptions are provided in accordance with individual student test data.
10) *Scheduling/resource allocation*: the capability to help coordinate student learning activities such as testing and instructor-student conferences, in addition to keeping track of the availability of learning resources; in general, to function in the role of a "scheduler".
11) *Grade contracting*: the capability to allow students to specify the content and level of achievement they wish to attain within predefined limits set by the instructor.
12) *Message handling*: the capability for students to leave messages for instructors and vice versa; a tactic for assuring the opportunity for student/teacher interaction on an *ad libitum* basis.

In a state-of-the-art assessment of CMI, Mitzel (1974) suggests that there are currently three levels of application:

> "In Level I the computer is used with instructor-input in *batch processing mode* to tabulate and accumulate systematic information about marks, attendance, test performance, etc. At this rudimentary support level, the role of the computer is chiefly one of replacing high quality clerical and tabulation services. In Level II the computer is still used with instructor-input in *batch processing mode* to accumulate records of student performance and *to feedback* these cumulative records to both the instructor and the student on a scheduled basis. A periodic prescriptive function, however, has been introduced at Level II advising students on a differential basis of what they need to do to remove various deficiencies which have arisen from the instructor's evaluation of their performance on the modules. . . . Level III of computer-managed instruction systems is characterized by a *real time interactive* interface between the learner and the computer, and by a diagnostic-prescriptive logic keyed to the student's responses to the material stored in the computer."

An examination of the foregoing list of functions which characterize CMI and the state-of-the-art assessment provided by Mitzel makes it clear that CMI is associated with two of the major current pedagogical concerns of the educational enterprise, namely *accountability* and the *individualization* of instruction.

(b) *Survey of significant developmental efforts*

The experimentation with various aspects of CMI has been very extensive in the United States in particular, and is currently accelerating at a pace that is difficult to assess. In the autumn of 1974, a national conference was held on the topic: "An Examination of the Short-Range Potential of Computer-Managed Instruction", and in the *Conference Proceedings* (Mitzel, 1974) a summary of 27 operational CMI systems is provided, nine of which deal partially or exclusively with mathematics. Among the projects that are reported, the one that has had the widest impact by far is the PLAN (Program for Learning in Accordance with Needs) Project which has a K-12 focus, embraces the language arts, science and social studies in addition to mathematics, and currently involves approximately 60,000 students representing 24 states and one country outside the United States. The system supports the following functions:

1) Identification and achievement-level assessment of each student.
2) Identification and recommendation of the quantity of instructional materials needed.
3) Daily status reports of each student and planning sections for teachers and students.
4) Daily instructional objective test-scoring.
5) Daily placement testing.
6) Daily PLAN achievement test scoring.
7) Periodic student progress reports.
8) Administrative reports.
9) Weekly summary reports.
10) On-line student progress reports.
11) History of student progress.
12) On-line ordering of additional materials.
13) Development of each student's programme of study.
14) Processing of locally developed objectives, independent activities, and courses.

In the United Kingdom, one CMI effort of note is the Hertfordshire Computer-Managed Mathematics Project under support from the National Development Programme in Computer-Assisted Learning. This project consists of a computer-managed system for teaching mathematics to students of mixed ability in the first two years of the comprehensive school. Altogether, about 4000 students and 12 schools are involved. The integrated course uses worksheets, videotaped materials and teletype terminals.

Another CMI undertaking of special interest is Project CAVA (11), an elaborate computer-managed correspondence course in basic mathematics developed and now in use in the Federal Republic of Germany.

(c) *Analysis/trends*

In a discussion of the need for computer-managed instruction, Allen (1975) has stated:

> "There are two major pedagogical concerns for both industrial and academic educational institutions: accountability and individualization. Accountability encompasses the need to demonstrate the effectiveness of educational programs in stimulating learning. It includes the need to define the behaviors to be acquired by the learner and the procedures through which the ability to perform those behaviors is measured. Record keeping and report generating mechanisms are additional needs to be dealt with in the attainment of educational accountability.
>
> Individualization, an important word in the modern educator's vocabulary, represents the recognition of human differences and the goal of providing the right educational program for each and every learner. Individualized education allows the student not only flexibility in the amount of time taken to acquire various behaviors and in the route taken toward behavior acquisition, but also in the selection of learning objectives.
>
> In individualized programs, problems in coordination arise from the heterogeneity of current student needs and activities, from the need for scheduling the use and availability of multi-media learning resources, and from the needs of learners and instructors to communicate effectively with each other at appropriate times. The need for tailored diagnostic testing, immediate test scoring, study prescription generation, and data security presents additional operational problems.
>
> The educational needs identified above can be met, simply in most cases, by CMI . . ."

It follows that, to the extent to which the concerns of accountability and individualization remain in the forefront of educational thought, CMI would be expected not only to become increasingly sophisticated but also to become more and more of a potent force in the conduct of classroom instruction and learning.

2.2.2. Computer-assisted instruction

(a) *Pedagogical characteristics and considerations*

The term "computer-assisted instruction" (CAI) as it is used here refers to the employment of a computer as a teaching machine, performing the functions of *tutor*, *tester* and *exerciser* (drill and practice). As has been pointed out by Nievergelt (1975):

> "The intellectual environment which gave rise to the first generation of CAI systems in the early 60's was strongly influenced by the programmed instruction movement, e.g. Skinner (1954). CAI was seen by many as a direct continuation of the mechanical teaching devices, Pressey (1926), with the processing and decision-making ability of computers finally providing the flexibility whose lack had denied success to the mechanical realizations.
>
> The dominant mood of optimism among many workers in CAI was rationalized by arguments along the following lines: 1) education is a labor-intensive activity, 2) technology applied to other labor-intensive areas in the past has increased cost-effectiveness drastically, 3) with programmed instruction as a teaching strategy and computers as a delivery device, a technology of instruction has finally arrived, and hence, 4) CAI will drastically improve education in the near future. 'Improve' had three aspects: 1) more effective instruction (learn better and faster), 2) cheaper instruction, and 3) overcome the shortage of teachers."

The pedagogical features of CAI, therefore, are essentially those attributed to programmed instruction by the futurists of twenty years ago. In its ideal forms, CAI would certainly seem to offer some unusual educational opportunities: to allow a student to proceed on an individual basis – in terms of content, pace and learning mode; to allow the educator to break down customary monolithic educational units which start and end at specified points in time and to which all students must conform; to provide the student with a highly interactive (responsive) learning environment that is capable of making "real time" (adaptive) adjustments in conformance with individual learning outcomes; and to permit the educator to engage in sophisticated forms of acquisition of student performance data and thereby remove much of the record-keeping routine that so often prevents a teacher from being able to attend to the actual business of teaching.

Unfortunately, the foregoing pedagogical potentials of CAI are not easy to "cash in" due primarily to the present inadequacy of the technology of programmed instruction. Moreover, due to the recency of the origins of the CAI movement (the early 1960s), there still is a dearth of "courseware" available for use in the schools, and the costs associated with the acquisition and operation of the powerful forms of the (CAI) technology which have been developed continue to be higher than those associated with conventional instruction.

(b) *Survey of major developmental efforts*

The United States is clearly the centre of the current, world-wide efforts in the CAI domain. This is not surprising in view of the fact that the concept of computer-assisted instruction originated in the United States, and that federal agencies there, particularly the National Science Foundation, have made heavy investments in pursuit of the development and exploration of this technology. Surely the most significant CAI project in the world is headquartered at the University of Illinois where the PLATO IV computer-based education system, and its predecessors, were developed in the Computer-Based Education Research Laboratory (12). The PLATO technology is now gradually finding its way out of the laboratory, however; as of 1 January 1975, for example, nearly 800 PLATO IV terminals were operating from 70 educational campuses including some 450 terminals at 60 locations other than the campus at the University of Illinois. The PLATO IV instructional system is capable of offering an extremely rich terminal learning environment, and in general, this technology has the potential for meeting the requirements necessary for widespread implementation in the schools provided that courseware of sufficient quality can be made available.

As might be expected, PLATO courseware development also has been vigorously pursued at the University of Illinois. Of special interest to this report is the Elementary Mathematics Curriculum Project which is now under way (13). This project is designed to provide roughly 150 hours of PLATO lessons for use by children in grades 4 through 6. The lessons will be concentrated into three strands: *graphing*, *whole number operations and concepts*, and *fractions*.

Another CAI effort of special interest that is now operational in the United States is the so-called TICCIT (Time Shared Interactive Computer Controlled Information Television) Project (14). As reported by A. R. Molnar (1975), this system ". . .provides services to 128 consoles simultaneously using a mini-computer and television technology. The terminal consists of a keyboard, a small color television to display computer generated displays, videotape or slides and audio". Two years of community college level mathematics have been developed for delivery by the TICCIT system, and these courses are now being tested in two community colleges.

Outside of the PLATO and TICCIT systems, most of the CAI courseware in mathematics that has been developed and had any substantial use to date in the United States falls into the *drill and practice* category. The two most widely known efforts along these lines are the "Stanford

Arithmetic Drills" (15) and the "Arithmetic Proficiency Training Program" (16), both of which were designed to be used as supplements in elementary classrooms.

The 1973 edition of the *Index to Computer-Based Learning* (Hoye and Wang, 1973) lists 374 programmes in mathematics though the expanded use of CAI for mathematics education makes it virtually impossible to identify all the developmental efforts currently under way. By and large, as indicated earlier, at the present time the United States seems to be the focal point for CAI curriculum development with close to 300 of the 374 programmes listed having been developed and implemented in schools there.

CAI developments in Canada are proceeding at the University of Western Ontario, Simon Fraser University, the University of Alberta and the *Laboratoire de Pedagogie Informatique*, to name a few. The Federal Republic of Germany has over 40 programmes developed by the University of Freiburg and the *Wissenschaftliches Institut für Rehabilitation und Beruflicher Bildung* in Heidelberg. Additonal research and development efforts in Europe have taken place at the University of Leeds, and the University of Glasgow, United Kingdom; the State University of Leyden, Holland; the *Conservatoire National des Arts et Métiers* in Paris; and the University of Liège in Belgium.

(c) *Analysis/trends*

Computer-Managed Instruction and *Computer-Assisted Instruction* constitute educational ideas that are still in their infancy, and thus the many analyses and prognostications that are now being put forward concerning them and their potentials for impact must be tempered by this fact.

In the CMI realm, however, the following assessment made by Scanlon (1974) appears to be justified:

> "Widespread use of CMI will occur if schools individualize instruction using structured curricula materials. Developing a computer management system is a fairly simple task if the curricula materials in use have well-defined behavioral objectives, alternative teaching methods, and criterion-referenced tests. Materials of this type are still not widely used in schools. A major obstacle to the use of CMI is the lack of published curricula materials which are highly structured."

In the CAI domain, the view expressed by Hooper and Toye (1975) likewise seems defensible:

> "Computer assisted learning is to be viewed as one of a range of media that are available to the teaching/learning system. Attempts to mount whole courses using just computer terminals, for example, are not likely to be acceptable either financially or philosophically. Computers should not be doing what can be done as effectively or more cheaply by other media. Development projects should be designed which exploit the characteristics of computer technology alongside other media such as teachers and books which have their own characteristics. The perspective is multi-media, with computer assisted learning being one of the media."

3 THE PAST, PRESENT AND FUTURE OF THE USE OF TECHNOLOGY IN MATHEMATICS TEACHING

Glaeser (19), in discussing the didactic of mathematics, has stated:

> "Four thousand years of blindness and deafness . . . (whereas) the history of mathematics, preceded by a long prehistory, extends over 4000 years, the history of the didactic of mathematics hardly covers one or two decades."

In light of the fact that the use of media in mathematics teaching is a didactical matter, Glaeser's comment provides a useful perspective from which to view the past, present and future of the newer educational technologies.

In any event, as Greenhill (1976) has pointed out, there are several noticeable trends in the development and uses of media. Greenhill's ideas may be described as follows:

(1) There has been a massive and continuing improvement in the "hardware" associated with the various technologies. A few years ago, projectors or video-tape recorders, for example, were operated by audio-visual specialists or by a few adventuresome teachers, but rarely by the learners themselves. All this has changed. Most the the hardware now being used has automatic controls and can easily be operated by students or teachers. Computers are also gradually becoming more accessible for day-to-day use in classrooms.

(2) Trends in the production of audio-visual materials have paralleled those of equipment. Films, filmstrips and slides, for example, are now being produced in great numbers to fulfil specific instructional needs.

(3) The applications of instructional television continue to grow and it is becoming commonplace for many countries to offer instructional programmes on a national basis. It is to be expected that the developing countries of the world will adopt such a practice.

(4) Methods of instruction are tending toward a greater emphasis on personalized learning, which, in turn, imposes the necessity for a much greater degree of flexibility. It is in this regard that technology seems to hold its greatest potential.

In discussing the report of the Carnegie Commission on Higher Education entitled *The Fourth Revolution* (1972), Molnar (1975) pointed out that the title of the report was derived from Eric Ashby's observation that four great revolutions have occurred in education. The *first* revolution took place when the task of education for the young was shifted in part from parents to teachers and from home to school; the *second* revolution was the adoption of the written word as a tool of education; the *third* revolution was the invention of printing and the widespread availability of books; and the *fourth* revolution was the development of electronics, notably radio, television and the computer, though the computer is distinctly the imperative in the fourth revolution. This is the idea of paramount importance in any analysis of the contemporary technologies; the computer with its vast potential for impact is in the centre of the stage.

It is difficult at this point in time, however, to make any dependable predictions about the exact forms that computer effects on education will take; there are even prospects of audio-visual communications networks that are computer controlled and the resulting potential for impact reaches far beyond the formal aspects of schooling. On this point, Molnar (1975) has reported that:

> "In the new town of Reston, Virginia, located just outside of Washington, the National Science Foundation has supported a project with the MITRE Corporation to demonstrate the feasibility of interactive, computer-controlled television systems for home use.
>
> The experiment has coupled a two-way cable television system with a computer to demonstrate the feasibility of using a standard television receiver for home computer driven displays. In addition to operating in a broadcast mode, you may communicate from your home to the studio or from the studio to your home through two-way television. The home user can also communicate with the computer through a touch-tone pad or terminal.
>
> The terminal may be used for calculations or answering questions displayed on the television screen. The demonstration included drill-and-practice lessons and a simple computing language

> for writing student originated programs. The system is ideal for the handicapped, or those with special learning problems or the elderly who might find it difficult or inconvenient to travel. It is foreseen that 2000 individual homes may be served at a nominal fee."

In discussing the possible role of computers in education, Nievergelt (1975) has said:

> "If one accepts the point of view that we are dealing with all aspects of interactive systems for education, rather than with a particular subset thereof that fits preconceived constraints, then I find it not too difficult to guess what the near future will hold in store, say the next 10 years. Two points stand out:
>
> - a rapid spread of sophisticated interactive systems in schools of all kinds, businesses, and later homes, to be used for education, information services, and increasingly, entertainment.
> - lesson material that is prepared without any particular ideology, theory, or self-imposed constraints, according to the motto 'anything goes that you can dream up and program'.
>
> If these two guesses are accepted as working hypotheses, then one can deduce almost with certainty that this field of interactive systems for education will have the following characteristics over the next decade:
>
> - computers with fancy terminals will be seen more and more as communications media which offer powerful possibilities not available in conventional media, such as newspapers, books, radio, television, films. Hence this field will develop its own profession or craft, of authors, producers or whatever they will be called."

The following statement, made by the late Ralph W. Gerard (1968), serves as a fitting conclusion to the line of argument that has been developed:

> "Computers, and computer-aided learning, as an important aspect of their use, are likely to make major changes in society, and in man himself. I alluded to this earlier: if the richer impingement from the environment on organisms able to receive richer stimuli has led to the growth of the brain, I strongly suspect that further enrichment of the environmental input may continue to accelerate that process. The real problem in society is the progressive complexity of the problems with which we have to deal because we're a more advanced system, a more advanced social organism, or epiorganism, as I like to call it; and we do have automation coming in and taking over the more menial jobs so that humans will have to be qualified to do the less menial ones. I think we haven't a chance of getting this done without CAI help. I doubt that anyone has remotely developed his inborn potentialities during his own life experience, formal education or otherwise. I would guess maybe only a third of the potential has been realized. I can't justify this figure but am pretty sure it's not wildly off, and some others agree with me. So one can right now upgrade the effective performance of men as they now are, but I foresee a change in man himself that is an input of a richer, more effectively structured learning experience. Just as the distance receptors and tools turned an ape into a superape, this richer input may well some day turn man into a superman."

It is clear that Gerard had great expectations for the educational benefits to be derived from the use of computers. Only time will tell, of course, but as Eric Ashby has observed, the computer is surely the technological imperative to the present and future in education.

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Resources

1. Films available from: General Learning Corporation, 3 East 54th Street, New York, NY, USA.
2. Film available from: International Film Bureau, 332 S. Michigan Avenue, Chicago, Illinois, USA.
3. Film available from: Modern Learning Aids, 16 Spear Street, San Francisco, California, USA.
4. Film available from: Moody Institute of Science, 1200 E. Washington Blvd., Whittier, California, USA.
5. For information contact the Mathematics Department, Carlton College, Northfield, Minnesota (USA), concerning the Computer Graphics for Learning Mathematics Project.
6. Project Director: J.W. Searl, Edinburgh University.
7. Project Director: J. Hunter, University of Glasgow.
8. Developed by the Department of Mathematics, University of Strathclyde, Glasgow.
9. For information, contact: Prof. Dr. W. Fraunholz, Institut für Mediendidaktik, Erziehungswissenschaftliche Hochschule, Rheinland-Pfalz, Federal Republic of Germany.
10. For information, contact: Educational Development Center, 39 Chapel Street, Newton, Massachusetts, USA.
11. For information, contact Professor Dr. Milos Lansky, Director, Institut für Bildungsinformatik Abteilung Mathematische Modelle und Strukturen, Paterborn, Tübingen, Federal Republic of Germany.
12. For information, contact: Computer-Based Education Research Laboratory, University of Illinois, Champaign/Urbana, Illinois, USA.
13. For information, contact: Dr. Robert Davis, University of Illinois, Champaign/Urbana, Illinois, USA.
14. For information, contact: John L. Volk, MITRE Corporation, McLean, Virginia, USA.

15. For information, contact: Dr. Patrick Suppes, Stanford University, Stanford, California, USA.
16. For information, contact: Science Research Associates, 259 E. Erie Street, Chicago, Illinois, USA.
17. For information, contact: Prof. Dr. Osvaldo Sangiorgi, Escola de Comunicacoes e Artes, University of Sao Paulo, Brazil.
18. For information, contact: National Film Board of Canada, 680 Fifth Avenue, New York, NY, USA.
19. Presentation made on "Les Origines de la Didactique des Mathématiques" at the Universities of Nancy and Strasbourg.

PANEL

J. Chastenet de Gery (France), M. Fiedler (Czechoslovakia),
W. Fraunholz (Federal Republic of Germany), J. Hunter (United Kingdom),
D.R. Lichtenberg (United States), M. Quere (France), S. Schuster (United States),
H. Stever (Federal Republic of Germany).

In addition to the panel members, L.P. Greenhill (United States), J. Nievergelt (United States) and D. Roseveare (United Kingdom) contributed to this chapter.

Chapter XII

The interaction between mathematics and other school subjects

H. O. Pollak

INTRODUCTION

The purpose of this chapter is to present an overview of the educational implications of the applications of mathematics and of current world-wide trends in this area. It is clear that this presupposes a structuring of applied mathematics itself. Therefore, section 1 presents an analysis of the meaning of applied mathematics – or rather its various meanings – and this will be followed in section 2 by a detailed examination of the various components of the definitions given in 1. Section 3 will use the framework developed in 1 and 2 to examine the effects of applications of mathematics on education and to see how the educational aspects of the relation of mathematics and other disciplines are changing. Finally, section 4 examines the forces that affect these changes, both those that help the teaching of the usefulness of mathematics and those that hinder it. Applications of one branch of mathematics to another within mathematics itself will not be considered in this chapter.

As will be seen in the discussion, the structuring of the ideas in this chapter has benefitted from the assistance of many individuals and of their published work. It is, however, fitting to note the work of Hans-Georg Steiner, and in particular his paper "What is Applied Mathematics?" (Steiner, 1974) from the Lyon Seminar on Goals and Means Regarding Applied Mathematics in School Teaching, for its influence on the present formulation. Many have contributed critical comments, and it is appropriate to single out Max Bell and Margaret Brown for their helpfulness.

1 THE DEFINITION OF APPLIED MATHEMATICS AND ITS VISUALIZATION

In discussions of applied mathematics, a large amount of unnecessary difficulty is sometimes created by differences in perception of the appropriate definition. These differences have come about quite naturally in recent years, since the variety of mathematics which has significant practical applications, the number of fields to which mathematics is applied, and the modes of applications have all undergone very rapid change. It is useful to think in terms of *four* different definitions.

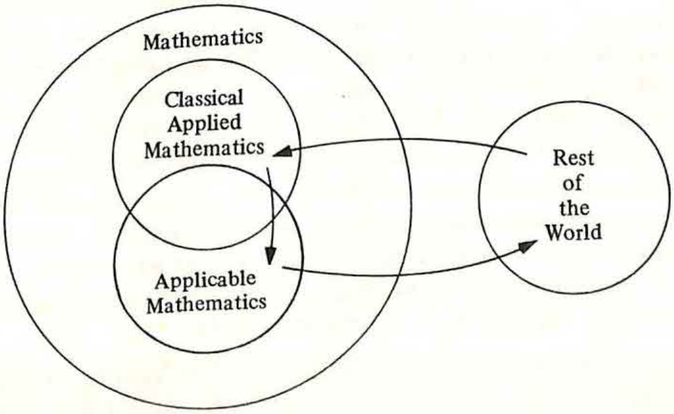
(1) *Applied mathematics means classical applied mathematics;* that is, the classical branches of analysis, including calculus, ordinary and partial differential equations, integral equations, the theory of functions as well as a number of related areas. It is sometimes convenient to annex those aspects of secondary mathematics which are essential prerequisites to calculus, in particular algebra, trigonometry and various versions of geometry. The fact that these branches of mathematics are the ones most applicable to classical physics is usually understood as part of this definition, but no actual connection with physical problems is implied.

(2) *Applied mathematics means all mathematics that has significant practical application.* This greatly enlarges the collection of mathematical disciplines included under (1). All the topics that have been considered world-wide for inclusion in the elementary and secondary school have significant practical applications – including sets and logic, functions, inequalities, linear algebra, modern algebra, probability, statistics and computing. Almost all the mathematics taught at the tertiary level (the undergraduate level at many universities) as well as much graduate mathematics are also included. In the views of many people, the most important areas of mathematics that are included in (2) but not in (1) are statistics, probability, linear algebra and computer science. There are many who feel that these topics are as important as classical analysis. Fields of potential applicability include more than physics, but, once again, only the mathematics itself is being considered.

(3) *Applied mathematics means beginning with a situation in some other field or in real life,* making a mathematical interpretation or model, doing mathematical work within that model, and applying the results to the original situation. Note that the other field is by no means restricted to lie in the physical sciences. In particular, applications in the biological sciences, the social sciences, and the management sciences have become extremely active. Many other areas of applications will also be considered.

(4) *Applied mathematics means what people who apply mathematics in their livelihood actually do.* This is like (3) but usually involves going around the loop between the rest of the world and the mathematics many times. An excellent example of the process involved in this definition of applied mathematics may be found in a report of the workings of the Oxford Seminar in the United Kingdom (Oxford, 1972).

A convenient aid in visualizing these four definitions is seen below:



In this picture the left-hand side shows mathematics as a whole, which contains two intersecting subsets we have called classical applied mathematics and applicable mathematics. Classical applied mathematics represents definition (1) and applicable mathematics, definition (2). Why doesn't (2) contain *all* of (1)? The overlap between these is great, but it is not true that all of classical applied mathematics is currently applicable mathematics. There is much work in the theory of ordinary and partial differential equations, for example, which is of great theoretical interest but has no applications which are visible at the moment. Such work is included in definition (1) as classical applied mathematics, since this contains all work in differential equations; on the other hand, if it is not currently applicable, it does not belong in definition (2).

The rest of the world includes all other disciplines of human endeavour as well as everyday life. An effort beginning in the rest of the world, going into mathematics and coming back again to the outside discipline belongs in definition (3). Definition (4) involves, as will be seen, going around the loop many times.

Other categorizations of applied mathematics have also been considered and can be examined in terms of the diagram. Typically, they involve a more detailed study of the process within mathematics itself than we shall undertake here. For example, applications of mathematics may consist of routine uses of mathematics, of mathematical reasoning as opposed to direct calculation, and of the building of models of various sizes going from small models through full mathematization of real situations to truly large-scale theories. Another very interesting way of slicing the pie may be found in Felix Browder (1976) "The relevance of mathematics". His categories consist of: (a) practical mathematics, that is mathematical practice in the common life of mankind in civilized societies; (b) technical mathematics, that is the use of mathematical techniques and concepts to formulate and solve problems in other intellectual disciplines; (c) mathematical research, that is the investigation of concepts, methods and problems of various mathematical disciplines including applied ones; and (d) mathematics as a universal pattern of knowledge, which means the science of significant form. His essay is highly recommended.

2 A DETAILED STUDY OF THE VARIOUS DEFINITIONS

2.1 The mathematics side of the diagram

The mathematical content of classical applied mathematics (definition (1)) and of applicable mathematics (definition (2)) have already been discussed. One recent trend has been the publication of books and articles showing the applicability of many of the mathematical disciplines which are not included in definition (1). To name just a few examples, Hans Freudenthal (1973) as well as M. Glaymann and Tamas Varga (1973) have written recent books on the applicability of probability; Tanur, Mosteller, Kruskal, Link, Pieters and Rising (1972) have edited a volume showing the great diversity of applications of statistics; R. H. Atkin (1974) in his book has included applications of topology, and Fred Roberts (1976) has devoted much space to applications of graphs and Markov chains. Journal articles are even more numerous; a few samples of particular interest follow – without the slightest pretence of coverage. Thus F. W. Sinden (1966) and Uwe Beck (1974) have shown some applications of topology; M. Dumont (1973) has discussed some uses of Boolean functions and J. H. Durran (1973) some applications of Markov chains. Recent applications of combinatorics and graph theory are examined, for example, by John Niman (1975), J. N. Kapur (1970) and W. F. Lunnon (1969).

A significant feature of applications of mathematics is that mathematical concepts and structures have important usefulness, not just mathematical technique. An interesting discussion of this point is given by H. G. Flegg (1974). Furthermore, since the relationship between mathematics and its applications is forever changing, there is a dynamic effect on mathematics itself. It has happened many times that areas of mathematics which were originally considered quite pure, and were developed with no thought of applications whatever, have turned out to be significantly useful. On the other hand, areas of mathematics which were invented only for application, with no thought of their possible contribution to core mathematics, have turned out to have an impact on pure mathematical disciplines. As an example of the former, the theory of entire functions has given notable insights in electrical communications; ideas of information theory, on the other hand, have been useful in such diverse fields as measure-preserving transformations and the theory of finite groups.

2.2 The rest of the world

Perhaps the outstanding feature of applications of mathematics in recent years is that the areas to which mathematics is applied have been increasing in number so rapidly. It is fair to say that no area of human endeavour is currently immune from quantitative reasoning or mathematical modelling. Besides the traditional physical sciences and engineering, the biological sciences, the social sciences, the management sciences, the humanities and everyday life are all arenas for interaction with mathematics. This is not meant to imply that mathematics is taking over all these other fields, but there *are* many interesting applications.

Perhaps the most extensive literature in recent years on applied mathematics from the point of view of the other disciplines has come in the biological sciences. An excellent overall survey appears in the book by J. Maynard Smith (1968). Books dealing with specific areas within the biological sciences include Victor Twersky (1967) on growth, decay and competition and R. M. May (1973) on the stability of ecosystems. Among the articles too numerous to summarize we note S. Karlin (1972a,b, the former jointly with M. Feldman) on genetics, S. P. Hastings (1975) on neurobiology, Arthur Engel (1971, 1975) and Beck (1975) on population models, W. D. Hamilton (1971) on the geometry of group behaviour, and several articles in "Computers in Higher Education" (1974) on the use of computers in biology. Not that new books and articles on mathematics in science have been lacking: We note particularly a little known volume by George Polya *Mathematical Methods in Science* (1963) as well as another portion of Victor Twersky (1967). Recent articles on mathematics in science include J. B. Griffiths (1976) on model building and mechanics, the conference report on "Modern Mathematics and the Teaching of Science" (1975), and the previously mentioned computer survey "Computers in Higher Education" (1974).

Another field which has recently flourished is the interaction of mathematics with the social sciences. Information on computers and statistics in the social sciences generally may be found in (Computers . . . , 1974) and (Teaching of statistics . . . , 1973); a fascinating and somewhat different viewpoint is represented in the article by H. R. Alker, Jr., "Computer simulations: Inelegant mathematics and worse social science?" (1974). The *Source book on Applications of Undergraduate Mathematics to the Social Sciences* (1977) contains descriptions of detailed mathematizations in many fields of the social sciences. To go on with specific fields, economics is extremely active for interactions with mathematics, although good expositions of the problems of model building in economics are not common. One nice example is "On the theory of interest" by David Gale (1973). Mathematical work in geography has also been quite popular in recent years, particularly in the United Kingdom. Again there are significant contributions in (Computers . . . , 1974) and (Source Book . . . , 1977), and an elementary treatment of weather forecasting in Durran (1973); see also King (1970). Mathematical psychology is represented by two recent survey articles by Anatol Rapoport (1976); *Source Book . . .* (1977) also contains extensive references to recent work. Besides their appearance in overall summaries, anthropology is represented by example in the book by L. Pospisil (1963) and the traditional mathematical theory of warfare by Arthur Engel in (1971). A magnificent example of mathematics applied to political science may be found in M. L. Balinski and H. P. Young (1975) "The quota method of apportionment". Mayer (1971) and Coxon (1970), for example, represent mathematical sociology.

The very large field of mathematical models in the management sciences including the entire area of operations research hardly needs description here. Sample articles of particular interest in recent years include those by F. J. Fay (1972), J. C. Herz (1973) and the delightful piece on mathematics applied to college presidency by J. G. Kemeny (1973). Mathematical models in medicine has been an increasingly active field; there is an excellent survey by J. S. Rustagi "Mathematical models in medicine" (1971). Mathematical linguistics has similarly become a major accepted field. Interesting particular articles appear, for example, as parts of Engel (1971) and *Source Book* . . . (1977), with Sankoff (1973) as another good source.

The penetration of mathematics into the humanities, including statistical and computer models, is a fairly recent event. Perhaps furthest advanced are mathematical analyses of art. We note, for example, A. V. Subnikov and V. A. Koptsik (1974) and a very valuable British summary of mathematical ideas and concepts in art by Beryl Fletcher (1976a). Mathematics applied to architecture is discussed by R. Fischler (1976) as well as in the summary work "Computers in Higher Education" (1974). Some examples of mathematical ideas in hobbies and handicrafts are given in Beryl Fletcher (1976b). Mathematical strategies for certain games such as NIM and the towers of Hanoi have long been familiar to, and enjoyed by, mathematicians. In recent years, there has been a great upswing in the discovery of optimal strategies for much more intricate games, and this has even provided one of the early applications of ideas from nonstandard analysis. We particularly note the work of E. R. Berlekamp and J. H. Conway, partly reported in Conway (1976). A nice example of optimal strategy for poker is given by W. H. Cutler (1975). Cryptanalysis has often been treated – see e.g. Sinkov (1968); for mathematics in sports see Klein (1972).

Besides the above-mentioned books and articles more or less devoted to specific areas of applications, there has been a trend in recent years towards the publication of excellent collections of articles and symposium reports which cover a broader spectrum. One of the earliest but still of great interest is the Utrecht colloquium "How to Teach Mathematics so as to be Useful" (Freudenthal, 1968). This was followed by the Echternach symposium "New Aspects of Mathematical Applications to School Level" (Echternach, 1973) and the Lyon seminar "Goals and Means Regarding Applied Mathematics in School Teaching" (Goals and Means . . . , 1974). Other noteworthy volumes of this kind include *Notes of Lectures on Mathematics in the Behavorial Sciences* edited by H. A. Selby (1973), *Topics in Behavorial Mathematics* by T. L. Saaty (1973), *A Source Book for Teachers and Students on Some Uses of Mathematics*, Max Bell (1967), *A Conference on the Applications of Undergraduate Mathematics* . . . (Knopp and Meyer, 1973) and *La Mathématique et ses Applications* by E. Galion (1972).

The preceding list well illustrates the current diversity of applications of mathematics in contrast with the historical monolith of applications to physics. It should not be assumed, however, that the arguments between those who stress the great variety of applications in recent years and those who feel that their total impact cannot compare to the 2000-year accumulation of success in mathematical physics have died down. In fact, this difference of instinctive value judgement underlies many of the arguments about mathematics education to which we will return later.

2.3 The model building process

When mathematics is actually applied to a situation in some other field, there are typically a number of distinguishable steps in the process. These consist of a recognition that a situation needs understanding, an attempt to formulate the situation in precise mathematical terms, mathematical work on the derived model, (frequently) numerical work to gain further insight into the results, and an evaluation of what has been learned in terms of the original external situation. This picture of the model building process has been widely accepted and there are many papers which elucidate the details from various points of view. Overall descriptions appear, for example, in the papers by M. S. Klamkin (1971), H. O. Pollak (1970) and P. L. Bhatnagar (1974). The same pattern, but applied specifically to operations research, appears in the paper by Gordon Raisbeck (1975) "Mathematicians in the practice of operations research"; its application to engineering may be found in A.C. Bajpai, L. R. Mustoe and D. Walker (1975), and again in the paper by H. G. Flegg (1974). M. E. Rayner (1973) in her paper "Mathematical applications in science" in the Echternach report describes in detail some of the difficulties in problem formulation. A quotation she gives from Eddington is particularly worth repeating, "The initial formulation of the problem is the most difficult part, as it is necessary to use one's brains all the time; afterwards, you can use mathematics instead". A proposal for better model building in mechanics is also given by J. B. Griffiths (1976). See also Wilder (1973).

The model building process has a number of interesting properties as well as pitfalls which we shall examine. A good model is one which is to some extent successful in explaining, or even predicting, external reality. If it fails to have this explanatory power then, no matter how satisfactory the mathematics itself, the model is not good applied mathematics and must be changed. This process can be quite painful for the mathematician but real progress in interdisciplinary efforts is often made through successive changes in the model. This is one of the reasons why definition (4) of applied mathematics involves going around the loop many times. Another phenomenon which sometimes happens is that a mathematical model predicts too much rather than too little. It may happen that phenomena observed in the other field are indeed explained satisfactorily, but that further logical implications of the model are not acceptable. For example, in the mathematics of communication a model of a signal which is of finite duration in time is very realistic. Similarly, a model of a communication signal using finite bandwidth comes up in many situations and gives quite satisfactory engineering results. Unfortunately, the two are contradictory and cannot be used at the same time in the same problem; models which do so unwittingly will lead to nonsense. On the other hand, attempts to understand this difficult situation fully have led to very interesting advances, see e.g. D. Slepian (1976).

Another feature of the model building process is that the purposes for which a mathematical model is created are also quite varied. In the physical sciences and engineering, the purpose is frequently very precise understanding which will in turn lead to action. In the social sciences, on the other hand, the purpose is often one of insight; you want to know whether a certain set of hypotheses could account for a particular observed phenomenon. It is often assumed, although not necessarily true, that these associations are in fact one-to-one correspondences. Physical models of why rivers meander, or why a rapidly slurped pice of spaghetti comes up and hits your nose, are not necessarily used for scientific decisions. On the other hand, mathematical models of shortest connecting networks and optimal pricing are often used for management action.

The overall picture of applications of mathematics would not be complete without a discussion of truly interdisciplinary activity. Much of the most exciting current work is in fact on the borderline between several fields, one of which being in the mathematical sciences. The above references will lead the reader to many examples of current interdisciplinary work.

3 EFFECTS OF APPLIED MATHEMATICS ON MATHEMATICS EDUCATION

3.1 Problems and problem solving in the schools

A framework for understanding the meaning of applied mathematics has now been established, and a number of ramifications of the various definitions have been examined. A look at effects of applied mathematics on education follows. It must be emphasized that many of the topics in this chapter represent ideas and experiments in various countries which cannot claim to be adopted on any large scale. Discussions at the Karlsruhe Congress did not bring forth any data which would substantiate broad use of applied mathematics in the schools.

Traditionally most of what was considered applied mathematics in the schools has been found under headings such as "word problems", "problem solving", etc. (This does not mean the "word problem" in the sense of modern algebra.) The meaning of such problem solving has been examined in a number of projects and articles in recent years. For example, the work of IOWO in the Netherlands is of particular importance. IOWO has also paid special attention to the differences in abstraction and precision between mathematical language and everyday language. The detailed meaning of problem solving is examined in papers by H. G. Flegg (1974), Beryl Fletcher (1976c) and H. O. Pollak (1969). Genuine applications of mathematics to other fields and to everyday life should ideally be in the character of definitions (3) and (4). It is often argued that a full presentation in the spirit of even definition (3) represents too large a project and takes too much time. In that case, the actual situation and numbers used in the word problem should at least be genuine extractions from an honest problem formulation. For example, estimates of crop yields and of times to complete a task should not be made to five significant figures, for this will never happen in real life. Too many plumbers in one room get in each other's way, and jobs are not always divisible. A current joint project of the National Council of Teachers of Mathematics and the Mathematical Association of America in the United States is producing a *Source Book* (1978) of hundreds of simple problems which are intended to be genuine in the above sense.

The opposite phenomenon is that the facts alleged in the statement of a problem are sometimes totally unreal. Problems which use wrong linguistics or impossible engineering or incorrect meteorology just to have some words from another discipline should be avoided. In this case, intent can nevertheless be important. Sometimes problems are clothed in a mantle of external vocabulary only for amusement, and the pretended application is not meant to be taken seriously. We shall call such problems whimsical problems. A strong argument in favor of such problems is made for example by Arthur Engel (1969) "Some examples are artificial, like fables. But just like fables, they have a moral, i.e., they facilitate insights into things that appear in the real world". For example, it can be quite effective to begin with an unsatisfactory oversimplification of a real situation, and to approach a genuine application in the sense of definition (4) through a series of increasingly realistic problems. Thus whimsical and unreal problems are not necessarily devoid of pedagogic value. However, if they are perceived as stupid, they may well be counterproductive. Similar discussions of real and unreal problems may be found in two particularly interesting papers by Margaret Brown (1972, 1973) and Mary Williams (1971). In particular, Mary Williams points out that the same difficulty of unreal models happens at a very advanced level as well as at the school level. See also section 1.1.5 of Chapter IV.

The increased awareness in many countries of the importance of teaching the applicability of mathematics has led to a number of very interesting attempts to collect real problems at various levels, and from various disciplines, and to make them available for teaching purposes. One collection at the school level (Source Book . . . Secondary School, 1978) has already been mentioned. Other general collections have been made by Max Bell (1972), Ben Noble (1967),

D. A. Quadling (1975), and C. W. Sloyer (1974). Collections devoted to particular disciplines, mainly at the university level, include the series on statistics by example (Mosteller et al., 1973), the social sciences problem book (Source Book . . . Social Sciences, 1977) and the collection of mathematical models in biology (Thrall et al., 1967), although the realism of problems in the latter collection varies. Another text in the same spirit, although it is organized as an actual course in engineering concepts, is *The Man Made World* (1971). It can be expected that very interesting collections of real problems in the above spirit will also be appearing in China. One such example of which we are aware contains, among other things, a number of excellent geometric problems from industry and agriculture (Applications . . . , 1975).

3.2 Mathematical subject matter in the schools

The diversity of applicable mathematics (definition (2)) which has emerged in recent years has greatly complicated the task of designing curricula for elementary and secondary schools. The traditional goals of preparing students for either shopkeeping or calculus (associated with definition (1)) cease to be uniquely valid when so many more areas in the mathematical sciences are of undeniable importance to so many of the world's people. As the number of reasonable choices increases, so does the difficulty of designing a curriculum. It has been argued by many that, for example, probability, statistics and computer science are as important for applications as the calculus. School materials for applications to many different disciplines have become available in recent years. Collections of materials involving applications to many different fields may be found, for example, in *Crossing Subject Boundaries* (Schools Council, 1970) and the materials from the Minnesota School Mathematics Center (Rosenbloom, 1963). The Chelsea Centre for Science Education project, "Science Uses Mathematics" (Chelsea) contains interesting applications to science which can be used in an interdisciplinary way, although this is not always done. *Applied Mathematics in the High School* by Max Schiffer (1963) also gives excellent examples of the relationship of mathematics and scientific applications from the point of view of the schools. A collection of examples which turn the tables and use physics to do mathematics has been made by Uspenskii (1961).

A major work examining curricular goals and pedagogy in the framework of an application to economics may be found in Damerow, Elwitz, Keitel, and Zimmer (1974). Biological applications may be found in Gibbons and Blofield (1971), and applications to geography in the materials by IOWO, in *New Ways in Geography* by J. P. Cole and N. J. Beynon (1968) and also in B. Fletcher (1976c). Applications to geography are also featured in the *Travaux d'Orléans (Les Mathématiques dans l'Enseignement . . .*, 1975), which in fact contains many other fascinating applications to a variety of fields throughout the curriculum, including economics, technology and medicine. This work also features references to recent work on applications in France and interesting philosophy on the usefulness of mathematics. An interesting application to political science may be found in Steiner (1966); environmental applications occur in the work of IOWO and in the book by Fred Roberts (1976). As we look at applications organized from the mathematical point of view, a superb collection of applications of linear algebra may be found in T. J. Fletcher (1972), and of statistics and probability in the work of Arthur Engel, e.g. (1970, 1973) and in *The Teaching of Probability and Statistics* edited by Råde (1970). *Mathematics Applicable* by the Schools Council (1975) also motivates much secondary mathematics through examples; the volume entitled *Logarithmic/Exponential* is a particularly interesting sample.

This great diversity of possible applications of mathematics, and of elementary branches of mathematics with significant applicability, has made the curriculum design problem very difficult. For example, topic A deserves to precede topic B in the curriculum if topic A is socially more important at this particular time, or if topic A is a prerequisite to topic B at this particular time.

As technology and social goals change, so should the ordering of importance. As available tools for teaching change, so will the order of prerequisites. These orderings will differ also from country to country. These facts make it even more difficult than it has been in the past to export curricula from one part of the world to another. Since an imported curriculum incorporates problems, situations and values which make no sense in a new country, this was probably never desirable, but is is even more questionable now.

3.3 The possible effect of applications on pedagogy

An appreciation for the different forms of applications of mathematics should affect not only the curriculum materials of the schools but also the pedagogy. If you examine even relatively simple uses of mathematics, you find that it is necessary to understand when and how and why the mathematics works in order to apply it correctly. There are several reasons for this. One is that mathematics which has been understood will be remembered better. Another more fundamental reason is the danger that mathematics which has been memorized without understanding will be misapplied. It is necessary to know where a particular method or formula comes from, exactly what kind of problem it will handle, and when and how it works in order to be sure that it will apply to a new situation. Curriculum reform in many countries has emphasized the "why" of mathematics in recent years on the grounds that it is essential for proper teaching of mathematics. What we see is that "why" is just as important for interactions of mathematics with other disciplines as it is for mathematics itself. The natural desire of mathematics teachers to emphasize understanding as well as technique is reinforced, not contradicted, by applications.

The model building process as developed through definitions (3) and (4) of applied mathematics interacts with mathematical pedagogy in a still deeper sense. Model building requires an understanding of the situation outside mathematics and of the process of mathematization as well as of the mathematics itself. You cannot hope to mathematize a situation without understanding it. Here we have yet another way in which "applied" problems which do nothing more than mouth words from another discipline are likely to mislead the student. A great weakness of some courses with titles like "Methods of Applied Mathematics" is that no attempt is made to provide an opportunity for the student to understand the situation and the mathematization process. This point has been particularly emphasized by H. G. Flegg (1974) and is further substantiated, especially from the point of view of future employment, in R. R. McLone (1973). Some of the college-level collections of real problems mentioned previously, for example Noble (1967) and *Source Book . . . Social Sciences* (1977), take particular pains towards the understanding of the situation in the real world.

Another pedagogic implication of the interaction between mathematics and other disciplines as it is described in definition (4) is that such interactions are clearly open-ended. Open-ended teaching of mathematics itself has long been recommended by mathematics educators in many countries, although adoption is rare. What does "open-ended" mean in this context? Besides the usual activities of solving problems and proving theorems, students should have the experience of finding their own problems to solve and their own theorems to prove. Such experience is an important factor in the mathematical development of the student. But exactly the same argument holds in the context of applications. It is very valuable for the student to have open-ended modelling experience, which besides its great pedagogic value is an accurate foretaste of mathematical applications in the real world. Experiments in open-ended discovery teaching of mathematical applications, many in the form of truly interdisciplinary materials, are under way in surprisingly many countries. An outstanding example is certainly China, where a major practical problem will be used for reference and inspiration throughout a course in calculus or linear algebra. There are many other examples of open-ended and truly interdisciplinary activities at the tertiary level, represented, for example, by the *Case Studies in Applied Mathematics* (1976), the books by T. J. Fletcher (1972), Maki and Thompson (1973) and Roberts (1976). At the elementary level, an outstanding example is provided by the USMES project in the United States (Lomon et al., 1975) in which students attack a series of action-oriented challenges by appropriate combinations of mathematics, science and social science. Truly open-ended applications are particularly difficult to introduce at the secondary level, and corresponding materials are very scarce.

3.4 Applications and teacher training

As mathematics teaching changes in the light of the increasing applicability of the subject, so should teacher training. Teachers should become familiar with the new fields of applicable mathematics, with the process of model building, and with the associated pedagogic emphases on understanding and open-endedness. There is a general tendency world-wide to reverse certain recent trends and to include more experiences involving applications in the training of prospective teachers. Perhaps the most exciting development in this direction is the pattern pioneered in the United Kingdom and now also spreading, for example, to Australia (Fensham and Davison, 1972), i.e. to make an internship in industry part of the training of a mathematics teacher. In this way, it is possible for the teacher to learn something of how the mathematical sciences are really applied. Practising teachers also sometimes help with the preparation of new interdisciplinary, open-ended materials (see e.g. Case Studies . . . , 1976). Especially in those countries in which there is currently an ample supply of teachers, those prepared in the broader mathematical sciences and familiar with applications of mathematics enjoy a stronger position in looking for employment. In other countries, applied mathematics in the sense of definition (1) has always been a strong component in teacher training, but experience with applications in the sense of definitions (3) and (4) has been missing. Once again, major industrial or agricultural experience has become part of teacher training in China.

3.5 Vocational education

A further educational effect of applications of mathematics is in vocational education. As the importance of the mathematical sciences increases for many disciplines, so does the need the workers and technicians in these disciplines to learn the most appropriate mathematical techniques. Noteworthy vocational materials in a variety of technical fields have been developed in a number of countries. For example, of the order of a dozen volumes of applications of mathematics in different technologies (clothing, carpentry, metal work, etc.) have been produced in Hungary. A different development in the same spirit is the increasing popularity of special curricula for technicians in computer science and data analysis. These have become particularly prevalent, for example, in the United States.

3.6 The implications of truly interdisciplinary teaching

Teaching which is truly multidisciplinary is very difficult to achieve at any level, but perhaps nearest to reality in the elementary school, where – in many countries – a single teacher normally handles most if not all subjects. The evidence for this may be found in the many multidisciplinary materials for the elementary school which have been mentioned. Such activities, when actually carried out in the schools, are especially satisfactory for students because they strengthen the relationship between school and real life. Students are not always satisfied with the promise of future gratification inherent in such statements as "you will find out why this is useful later on", and are pleased with the applicability of mathematics to problems in which they are interested. This is particularly stressed, for example, by IOWO and USMES (Education Development Center, 1974, 1975). However, if the time during the school day is apportioned according to disciplines, it is necessary that the time for multidisciplinary activities be contributed by the various disciplines involved. This implies, at a minimum, that multidisciplinary projects must state what responsibility they will take for specific topics in the several disciplines. Appropriate teacher training at the elementary level is very necessary. On the secondary level, the implications for the structure of the educational system are much more severe. If a single unit involves mathematics, science, social science and language arts all in a significant way, who is going to teach the material, who will contribute the time, how should the school be organized? These problems have not been solved, although team teaching is one possibility; see also Rao (1975). They are discussed particularly in section 3.7 of Chapter III and in the *Report of the Memphis Conference* (Education Development Center, 1974). At the university level, multidisciplinary educational activities may take the form, for example, of genuine model building courses discussed previously, or of team teaching by faculty from mathematics and from a field of application of sections in basic courses such as calculus, linear algebra, and statistics. An example of a master's programme with multidisciplinary experience is Hunter College (1974).

4 THE FORCES THAT AFFECT CHANGE IN THE INTERACTION BETWEEN MATHEMATICS AND OTHER DISCIPLINES

4.1 Forces that help applications of mathematics

The drive to make mathematics in the school more applied is world-wide. As we examine some of the factors which tend to help or hinder this drive, we are in fact making inductions from the world-wide literature on applications of mathematics which we have been discussing. Perhaps the strongest force that can aid the applied point of view is a recognition that in the teaching of mathematics the applied viewpoint is a more accurate reflection of the country's social system than some alternate guiding spirit. In some countries, the problem is not stated as crisply as that, but the tendency towards making mathematics more applied is explained as a reaction against something called the "new math". That this reaction can hurt as well as help applications of mathematics will be seen in section 4.2 below. More specific forces that help the tendency towards applications in the teaching of mathematics are many:

(a) The increased mathematization of many other fields has, as we have seen, made really interesting applications of mathematics increasingly available. Mathematics teachers are, after all, always happy to use a good problem when it comes their way.

(b) There is also some increasing realization that applications are an integral part of mathematics teaching. The mathematics curriculum has, in the opinion of many educators, not completed its task if the student has not participated in, or even experienced, genuine applications of mathematics.

(c) The problems of motivating students become easier when applications are used as one of the possible motivations. There are many different ways in which any particular mathematical topic can be approached; it is not to be expected that a single approach will be best with all students. If greater individualization of mathematics instruction is desirable, it becomes increasingly useful to have available, approaches to a particular topic through applications.

(d) The recognition that the job opportunities for mathematics students at all levels are increased when the student knows how mathematics is applied has also been very helpful. This same argument may also apply to prospective mathematics teachers.

4.2 Forces that hinder applications of mathematics

Forces that hinder applications of mathematics in teaching have perhaps been discussed more widely than those that help. Two references of particular interest are the International Conference on Physics Education (Walton, 1975) and the article by A.D. Turner (1966). The difficulties are partly on the mathematics side and partly with the other discipline:

(a) Some mathematicians fundamentally believe in the purity of mathematics, and do not, in their view, wish to corrupt the beauty of mathematics through extraneous considerations. A different way of phrasing this idea is that the pure mathematics itself is very economical, in that an individual mathematical idea can be the basis of many different ideas useful in different disciplines. Thus, it is argued, the basic mathematical idea should be taught centrally to all students, and individual applications tailored to particular groups of students elsewhere.

(b) Some mathematics teachers are perhaps ignorant of other disciplines and may even be afraid of them. The training of mathematics teachers in many countries includes little attention to applications, and even less to applications of the mathematics they will be teaching. Mathematics teaching has been a possible way of earning a livelihood while having relatively little contact with the real world. When an effort is made to change this situation, it may be resisted by mathematics teachers of this kind.

(c) The reaction against the "new math" might at first glance be expected to be favourable to applications, but this is not necessarily the case. First of all, many aspects of recent curriculum reform in mathematics, such as the expansion of relevant subject matter, the emphasis of understanding, and the desirability of open-ended teaching, are aspects which are really favourable to applications of mathematics but which might again be lost. Furthermore there is a current popular urge to return mathematics education to "basics". Unfortunately, this is sometimes accompanied by a belief that an emphasis on basics is incompatible with higher cognitive strategies, one of which might be relating mathematics to the rest of the world.

(d) A further difficulty, as we have seen, is that time for applied mathematical activities must be carved from the rest of mathematics or from other subjects, and there will always be those who resist such changes.

Other forces which hinder applications of mathematics are concerned more specifically with making the connection with another discipline:

(e) Many teachers and practitioners of these other disciplines do not themselves use much mathematics, and have not kept up with the recent analytical developments in their fields. Such people may resist new and more powerful mathematical ideas, and compact notation, even though they have the inherent capacity to benefit the applied discipline in quite specific ways.

(f) Next, other disciplines may sometimes use mathematics in a way which differs from the way in which it has been taught in mathematics classes. There are questions such as the following: Do you keep units throughout a series of mathematical equations? Do you expect to make an interpretation in the field of application of intermediate steps? Should the independent variable in a function be made explicit – e.g. when is velocity a function of time and when of position? Should we use matrices or multiple subscripts? Such differences in point of view, notation and specific technique make communication difficult.

(g) Another aspect of the same problem, emphasized in Turner (1966), is that meaningful science is best taught by activity. Activity is likely to breed dirty numbers, and dirty numbers are not used in the teaching of mathematics; hence the kids cannot handle them. The popularization of the hand-held calculator is very likely to help remove this problem. The calculator facilitates concentration on the reasons and methods for applying mathematics in a particular situation.

(h) Finally, there has been in some countries a marked decline of student interest in science. Such a development may deprive mathematics teaching of some of the most powerful applications of mathematics.

The long range answer to this last mentioned problem is that interesting applications of mathematics occur in all fields of human endeavour, not just in science. The world-wide movement to capitalize on applications of mathematics in the broader sense has been the subject of this chapter.

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PANEL

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Chapter XIII

The role of algorithms and computers in teaching mathematics at school

A. Engel

1 COMPUTER USE AT SCHOOL: PRESENT STATE AND CRITICAL APPRAISAL

The main purpose of this chapter is to give guidelines for computer use in *mathematics teaching.* But first we give a brief survey of the current state of computer use.

We start with the United States. In 1970 and 1975, the American Institute for Research (AIR) conducted two surveys of public secondary schools, i.e. schools having at least one of the grades 9 to 12. These reports by Bukoski and Korotkin (1975) and Darby (1970) provide reliable data of the *extent* and *type* of computer use as well as trends for a five-year period. Let us look at some of the major findings:

(a) *The language problem.* BASIC has become the predominant computer language. The frequency of use of the three most popular languages is shown by the following table:

	1970	**1975**
BASIC	25.8%	62.4%
FORTRAN	33.6%	18.6%
COBOL	3.4%	4.6%

The rise in COBOL is due to business education classes and administrative computing. Many computer scientists, especially in Europe, are hostile to BASIC. Indeed, BASIC is inadequate for computer science. For mathematics BASIC is satisfactory, especially its latest versions which include the control structures *if-then-else*, *while-do* and *repeat-until*.

(b) *Extent and type of computer use.* In 1975, the computer was used by 58.2% of the schools compared with 34.4% in 1970. If the current growth rate continues, by 1984 every secondary school in the United States will have access to a computer. These are quite impressive numbers, but on closer scrutiny some sobering facts emerge. Of the user schools, a total of 54.1% were using the computer *only* in administration. Thus only 26.7% of the schools reported some instructional application; 43.2% of the instructional computer courses were in mathematics. These numbers are still respectable, but let us look at the potential number of students involved in a typical user school (average population 1000). There were on the average five terminals per school. This is not bad, but again this number is deceptive since wealth is usually concentrated in the hands of a few. The *mode,* i.e. the most frequent number of terminals is *one*. The *median* is *two.* Thus a typical school has one or two terminals. Lack of terminals is currently one of the biggest bottle-necks. Student access is very restricted and this severely limits individual exploration. Another problem is the narrow base of educational computing. Usually only *one* or *two* mathematics teachers are involved. Thus the actual number of students using the computer is quite modest and the depth of usage is mostly superficial.

There is a lack of competent teachers and scarcity of good printed material. The majority of schools are operating within an information vacuum. Many schools are still "reinventing the wheel". This is certainly a valuable learning experience, but it may be wasteful in terms of time, money, and personnel.

The computer tends to generate a new level of demand by students and teachers. In order to survive and to thrive, instructional computing requires increasing fundings. In reality the relative amount of the school budget spent on computing has not increased since 1970.

Let us now turn to western Europe and Japan. Here the total amount of computing activities is smaller than in the United States. In most of these countries (programmable) desk-top calculators dominate. In the United States, only 6.7 per cent of the user schools rely solely on desk-top calculators. But this rather modest share is increasing as the calculators become cheaper, more powerful and versatile. At the moment, one should not buy a calculator if it does not offer a language at least as powerful as BASIC. Calculators with obscure machine languages should be phased out.

The Japanese government has decided to provide each high school in the country with one desk-top calculator. The project started in 1970 and will be completed by 1980.

In the Federal Republic of Germany, between 5 and 10 per cent of the high schools possess a desk-top calculator. They are acquired through the initiative of dedicated mathematics teachers. Usually part of the money comes from donations of parents.

In the Federal Republic of Germany and some other European countries, there is a strong trend towards introducing informatics (computer science) as a new school subject. Detailed goals and contents of a secondary school course in informatics can be found in (Zielsetzungen . . . , 1976). There are several informatics textbooks for this level in existence, the most remarkable and ambitious is written by a top computer scientist (Bauer and Weinhart, 1974). There exists one good informatics textbook which is written specifically for secondary school teachers (Claus, 1975). The textbook *Computerkunde* (Görts et al.) is used in Dutch schools for computer appreciation courses.

The "Mercouroff Plan" in France has the opposite tendency (Mercouroff, 1975). Let us look at the idea behind it and the actions taken so far: It is not desirable to introduce another subject (informatics) into an already overcrowded curriculum. Informatics should not be considered as an independent discipline, but as a mode of thought and problem analysis (algorithms, flowcharts), which is relevant to all subjects taught in school. Between 1970 and 1975, some 430 secondary school teachers of all subjects were given one year of full time training in informatics. In addition, about 5000 teachers received some training by correspondence courses. It is hoped that each teacher will use computer science methods in his subject. The further expansion of this national experiment has been stopped for financial reasons.

In France, large scale experiments on the use of desk-top calculators are being conducted by the IREMs (*Instituts de Recherche sur l'Enseignement des Mathématiques)* and the INRDP

(Institut National de Recherche et de Documentation Pédagogiques). About 200 teachers have been engaged in this project since 1971, and the results are published in two comprehensive surveys (Emploi . . . , 1972 and 1975). In addition, some of the IREMs provide regular informatics training for mathematics teachers.

In some European countries, there is a trend toward a special computer language for schools: ELAN in the Federal Republic of Germany, E.S.L. in France, CESIL in the United Kingdom, ECOL in the Netherlands.

There are two major uses of computer education with opposite philosophies: *CAI (Computer Assisted Instruction)* and *problem solving*. The proponents of CAI use *the computer as a teacher*. The advocates of problem solving use *the computer as a pupil*. In CAI, *the computer controls the student*, in problem solving, *the student is in control of the computer*. Early CAI was programmed instruction using the computer as a teaching machine. Between 1960 and 1970, immense effort was put into CAI. However, the results were disappointing. The future looks somewhat brighter due to two significant CAI-projects: PLATO (University of Illinois) and TICCIT (MITRE-Corporation); see p. 226 above, or Nievergelt (in Lecarme and Lewis, 1975).

The philosophy of problem solvers is as follows: The best way to learn something is to teach it. The computer plays the role of a "model student". It is a very demanding student since it forces you to express the topic to be taught as an algorithm. By doing this, the topic becomes completely clear to you. If the computer can understand you, then you can be reasonably sure that you have grasped the idea you want to communicate. This point of view is well expressed by Kurtz (1970) and Luehrmann (1972), both of Dartmouth. The AIR report gives data on types of computer application for each subject area. For mathematics, the percentages of classes making use of various applications is as follows: CAI: 59%; problem solving: 62.3%; computer science: 29.6%; gaming and simulation 30.6%; CMI: 48.8%. The relative amount of time spent on these applications was not determined. In Europe, the use of CAI is negligible.

Most problem solvers abhor the CAI approach as a machine-dominated, passivity-inducing, uncreative, preordained and premasticated instruction. This is a reaction to the early and trivial uses of CAI with exaggerated claims; see Suppes (1966) for such a grandiose claim.

In this chapter, we ignor CAI since it has been dealt with in section 2.2 of Chapter XI. We will concentrate exclusively on problem solving. Most of the creative uses of computers belong to this area.

A very active proponent of student controlled computing is S. Papert from the Artificial Intelligence Laboratory of MIT. Since 1969, he has experimented with elementary school children, using the simple and powerful language LOGO. His papers (Papert, 1971, 1972a-d) contain many original ideas which go far beyond mathematics teaching. Papert contends that a child cannot learn much from a CAI program, but it can learn very much by writing CAI programs. The best way to teach a boring subject like arithmetic to a child is not by drill, but by asking it to write a program which teaches the subject. He reports that children can get passionately involved in writing teaching programs.

T.A. Dwyer, the director of the SOLO-Project at the University of Pittsburgh, tries to combine the two uses of a computer as a teacher and as a student. Borrowing the terminology from flight instruction Dwyer calls the two uses "dual mode" and "solo mode". In the first, the computer is in control; in the second, the student is in control. Dwyer has worked with high school students since 1969. He has developed many ideas of how to integrate computing into the mathematics curriculum, see Dwyer (1971, 1974, 1975a-b). *A Computer Resource Book: Algebra* by Dwyer and Critchfield (1975) can be used as a supplement to the traditional high school algebra course.

In this book, CAI programs dominate. It is not the use of CAI programs but rather the writing of teaching programs and the analysis of ready-made CAI programs. The aim is to teach algebra. As a by-product, the student becomes a proficient programmer.

The Learning Research Group (LRG) at the XEROX Palo Alto Research Center is developing a small personal computer of great power and flexibility. It uses SMALLTALK, a simulation language for expressing and viewing dynamic processes. The system has a graphic display screen and can be used for problem solving, music, painting, animation, physical simulations, file systems, etc. The group is working with children from ages 8 to 15 and it is influenced by the ideas of Papert and Feurzeig; see Goldberg and Tenenbaum (1975) and especially Goldberg et al. (1976).

The CAMP series of textbooks for grades 7 to 12 by Johnson et al. (1968-70) represents a typical use of the computer and computer programming in a traditional mathematics course.

Finally we want to point out the main danger in computer oriented mathematics courses. Such a course may degenerate into a course in computer programming with mathematics achieving secondary importance. The most critical problem world-wide is teacher education. Teachers at all levels must be educated in the use of computers in the classroom and especially in their own subjects. The whole volume (Milner, 1975) and part of (Colman and Lorton, 1976) is devoted to the state-of-the-art discussion of teacher education in the instructional use of computers. There is a critical shortage of books which show how to use the computer in mathematics teaching. Nievergelt et al. (1974) have written such a book on a high level; it grew out of courses for secondary school mathematics teachers. Engel's book (1977) surveys the whole of school mathematics from an algorithmic point of view. It contains hundreds of examples and problems from the area of school mathematics. Many of the algorithms can be executed with the pocket calculator. The book grew out of courses in computer oriented mathematics for junior high school teachers. Section 5 of this chapter is an excerpt from this book.

Textbooks are beginning to appear which try to integrate computing into standard college courses such as finite mathematics, probability, and calculus. Typical examples are Dorn and McCracken (1976); Kemeny, Snell and Thompson (1974); Lax et al. (1976); and Snell (1975).

To summarize, we can say that by necessity computer use at school is in an embryonic stage. It consists at the moment of an accelerated use of the methodologies of the past. The overthrow of antiquated concepts and methods has not yet started.

2 ALGORITHMS AT SCHOOL: PAST AND PRESENT

Algorithms have always played a major role in school mathematics, yet the lack of an efficient tool for executing algorithms prevented a consciously algorithmic attitude. Before the advent of the computer, it was not recognized that the construction of algorithms is an interesting problem which is an excellent training ground for precise thinking and thus an indispensable part of general education.

Children at school were often treated as small programmable calculators who had to store algorithms in their heads and execute them on demand. For this reason, algorithms fell into disrepute as mindless rules which have to be memorized without comprehending them. This negative attitude toward algorithms still persists. Lack of an algorithmic attitude on the part of the teacher prevents a sensible use of the computer. The diffusion of an algorithmic style of thinking among the teachers is the really decisive issue.

A substantial part of school mathematics must be restructured from an algorithmic standpoint. This can be done even in developing countries where computers are not available. Pocket calculators could be used to initiate a successful reorientation toward algorithms.

There are many people throughout the world who propagate the computer (the device) with missionary zeal. Among mathematics educators, there is hardly anyone who beats the drum for algorithms. This is in spite of the fact that leading computer scientists argue in a convincing way that algorithms are the core of computer science; see Knuth (1968-1973, 1974) and Aho et al. (1974). We should try to initiate a new trend by getting the algorithmic bandwagon rolling. For this reason, the main part of our report deals with algorithms and not computers.

3 THE ROLE OF THE POCKET CALCULATOR

The sudden emergence of the pocket calculator and its unprecedented success pose a formidable challenge. In the near future, the influence of this miraculous tool will be much greater than that of the computer. Its best-selling status indicates that people prefer their own computer just as they prefer the convenience of their own car. Maybe the future development will tend toward small and versatile personal computers like the Dynabook being developed by the XEROX research group (Goldberg et al., 1976), if they can be produced at an acceptable price, for instance the price of a colour television set.

We will list some problems posed by the pocket calculator. More details can be found in the report by M.N. Suydam (1976); appendix F of the report, written by M. Bell and Z. Usiskin, gives a deep investigation of the probable impact of the calculator on school arithmetic.

(a) The teacher needs quick help. He does not have time to adapt slowly to the calculator. Students bring a variety of models to school and teachers must cope with the situation immediately. The teacher must be well informed about old and new models, and he should be able to advise parents who want to buy a calculator.

(b) Some people think that we can save hundreds of hours of arithmetic drill and use this time more creatively by giving more attention to basic ideas and algorithms. Everybody will enjoy mathematics even more since the calculator eliminates the least pleasant aspects of doing mathematics.

However, in a good arithmetic class, children investigate mathematical situations. Arithmetic drill is acquired as a by-product of these investigations. In such a class the time saved by calculators is not substantial. Moreover we cannot discount the possibility that arithmetic drill somehow contributes to the learning of important mathematical concepts. In addition, drill may contribute substantially to the familiarity with numbers and the development of numerical intuition.

(c) Most educators point out that children must still acquire basic numeracy. But what does this numeracy include?

Children should still be able to operate with small numbers, understand place value, and make rough estimates with big numbers. The calculator also demands more approximations, order of magnitude, scientific notation, roundoff error, significant digits, floating point and fixed point notation, absolute and relative error. Some of these topics could be postponed to junior high school.

The suggestion to drop completely the multiplication and division algorithms in elementary school is probably too radical. But we should work with smaller numbers. Multiplication (division) by a multiplier (divisor) of one or two significant digits may be enough.

(d) Some people think the calculator will help children acquire basic numeracy, while others think it will do harm or even prevent them from attaining it. To maintain facility in hand calculation requires constant practice, which will be lacking if calculators are available. Thus hand calculation may be a dying art which should be left to die quietly.

(e) The calculator requires an early introduction of decimals. They should come before fractions rather than after. Should we postpone common fractions and operations with them for the algebra class?

(f) Outdated computational techniques with logarithms and the slide rule should be dropped without delay.

Do we still need mathematical tables? If yes, what should they contain? Presumably formulas and fundamental algorithms.

(g) The calculator makes it possible to bring into school some new and important topics which were avoided in the past because of computational difficulties.

(h) In elementary grades, the four-function calculator is completely adequate. It should have at least one memory register. Two or three memory registers would be very helpful, as can be seen by looking into the curriculum outline given in section 5 below.

In junior high school the seven-function calculator is required:

$$+\ ,\ -\ ,\ \times\ ,\ \div\ ,\ x^2\ ,\ \sqrt{x}\ ,\ 1/x$$

For senior high school and college we need in addition exp, ln, sin, cos, tan, arcsin, arccos, arctan. It is probably best if the student starts in elementary school with a seven-function machine and switches in senior high school to a fifteen-function machine.

(i) What is the average life time of a calculator? Four to five years? We do not know.

(k) A curriculum designed for the currently available calculators is not viable. Because of the rapid advance of technology the difference between pocket calculators, desk-top calculators and computers will gradually disappear. In 10 to 15 years everybody may have access to a powerful and flexible personal computer. The pocket calculator may well be a transient phenomenon.

(l) The pocket calculator is a powerful tool for numerical mathematics but a poor data processor. It cannot handle arrays. Small size is incompatible with the rapid input-output requirements of data processing.

(m) Personal ownership means unlimited accessibility. This makes the machine your slave, not vice versa. The pocket calculator changes the attitude of his owner. He finds himself using the calculator with increasing frequency. As he gains mastery of his powerful helper his demands on it will increase dramatically. It is his problem solver, educator and entertainer.

(n) What is usually attributed to lack of intellectual ability in children may be due in part to lack of interest in arithmetic drudgery.

(o) A substantial part of what is done by teachers with a terminal can be done as well and sometimes better with a pocket calculator. For traditional mathematics instruction the pocket calculator is quite adequate. Its slight drawbacks compared to a computer are balanced by advantages such as unlimited accessibility. It also forces us to use *efficient* numerical algorithms which are also more instructive. The search for efficient algorithms is a powerful motive for a deeper study of the properties of mathematical functions.

Teachers often believe that the tremendeous speed of a computer decreases the importance of efficient algorithms. For non-numerical (combinatorial) algorithms, just the opposite is true.

(p) With time consuming computations out of the way, we can give more attention to basic ideas. The definition of mathematical concepts will be influenced by the pocket calculator. So will algebraic manipulations. We will learn to transform algebraic expressions in such a way that they are easy to evaluate on the pocket calculator. We may prefer nested parenthetical form which permits evaluation on a memoriless calculator without using a scratch pad. Thus

$$1 - x^2/2 + x^4/24 - x^6/720$$

could be transformed into

$$1 - x^2(360 + x^2(x^2 - 30))/720$$

Some formerly difficult topics become trivial, such as compound interest or solving higher degree polynomial equations.

4 SCHOOL MATHEMATICS FROM AN ALGORITHMIC STANDPOINT

We propose guidelines for mathematics teaching. These are applicable to most countries, and they are not sensitive to the rapid evolution of computer technology. They can be implemented if there are some pocket calculators in the classroom. They imply a slight change in content and a major change in the point of view. The guidelines are flexible, so that you can make small or sweeping changes in your classroom. The guidelines can be summarized into one sentence:

Teach mathematics from an algorithmic standpoint

At the beginning of our century, Felix Klein initiated a reform of mathematics teaching. The reform movement adopted the slogan "functional thinking". The reformers claimed that functional thinking must pervade all of mathematics teaching. What the student should have learned in his mathematics classes is thinking in terms of functions. This reform has profoundly changed school mathematics.

The time has come for a new reform with the slogan "algorithmic thinking". Algorithmic thinking should pervade all of mathematics teaching. Teachers must realize that a major activity in the classroom should be the design and analysis of algorithms. Many concepts can be introduced constructively by programs. Each program is an embodiment of a concept, a function, a mathematical object. Programs are abstract machines that can be manipulated and their outputs can be studied for various inputs. By observing the structure and behaviour of a program, the student actively learns the concept which is represented by the program. The interaction between pupil and program is decisive for the learning process. Students must learn to modify programs and to construct composite programs out of simpler programs. Clarity and simplicity are the hallmarks of good style and good mathematics. Algorithms should be lucid. This holds whether they are expressed by flowcharts or statements. Initially both kinds of representations will balance. One should slowly reduce flowcharting, which is a primitive and clumsy tool. By using it too much, it becomes addictive.

First, algorithms will be stated in the native language of the child. This language is standardized step by step. Finally, we wind up with a restricted vocabulary (if-then-else, while-do, do-until, etc) which can be translated into any computer language in a straightforward manner. This vocabulary is like a mechanical engineering kit with whose parts one can construct many different gadgets (abstract machines or algorithms). The purpose of computers is to drive these machines as it is the purpose of electric current to drive a vacuum cleaner.

Reading and writing (composing an essay) in the mother tongue are equally important. Children acquire a good writing style by reading simple prose written in a lucid style (study of literature). Similarly, reading and writing programs should be equally stressed. The student acquires a good programming style by reading many programs written in good style (study of the literature of published algorithms). In primary grades, reading and executing algorithms and finding out what they are doing will predominate. Writing a sophisticated program, like writing a good essay, requires some maturity and organizational skill.

For every function f that we study in school, we should give at least one algorithm for its computation. To associate a concrete meaning to f the student must know an algorithm which for input x produces f(x) *without tears*. Preferably one should have several algorithms for the same function. Then you can compare them: which is more efficient, more lucid, more elementary, more universal.

5 A MATHEMATICS CURRICULUM FROM AN ALGORITHMIC STANDPOINT – OUTLINE OF TOPICS

In this final section, we identify the basic algorithms occuring in school mathematics, grades 5 to 12 (or 13). Most of the algorithms can be executed with a pocket calculator. For a comprehensive treatment see Engel (1977). At school it may be preferable to use "procedure" instead of "algorithm". We use a simple language which can be understood without explanation. A higher level language would be preferable, but this would require more space. At the outset we make some remarks about *numerical mathematics*. The computer or calculator is a powerful tool for numerical experimentation. For instance, the speeding up of numerical processes by extrapolation leading to the Romberg scheme (see section 5.5) is easy to discover by numerical experimentation. The Romberg Scheme requires pocket calculators with at least four memory registers (see section 5.5). Error analysis should be done by test cases; – run the algorithm for special inputs with predictable outputs. For example:

(a) Apply the $\log_{10}$-algorithm to $\sqrt[3]{10}$. Comparing the output with 0.333... reveals the accuracy of the algorithm.

(b) Apply the sine - algorithm to $\pi/2$, $\pi/3$, $\pi/6$.

Numerical common sense should be stressed through all grades. We must identify the basic ideas of numerical mathematics that should be familiar to every student using a computer or calculator: fixed point and floating point notation, absolute and relative error, overflow and underflow, roundoff error and truncation error (an infinite process is replaced by a finite one); subtractive and multiplicative cancellation as in $\sqrt{n+1} - \sqrt{n}$, and $(1 + 1/n)^n$ for large n (- how do you avoid cancellation?); numerical stability of a feedback loop (- will a small error grow or decay through successive iterations?).

Tables of mathematical functions should die quietly. Table look up is more costly than recomputation by an efficient algorithm. Instead of information retrieval, we prefer information regeneration.

The following part of this chapter presents a topical outline of a mathematics curriculum from an algorithmic standpoint. This outline includes more topics than would normally be feasible with any one class. For the reader's convenience, a list of contents of the remaining part of the chapter is given below.

Comprehensive outline of topics

5.1 Program reading and program writing for lower grades

- 5.1.1 Some simple, basic procedures
- 5.1.2 Definition of functions by statements and flowcharts
- 5.1.3 Ancient Egyptian multiplication procedure
- 5.1.4 Growth, decay and the fast powering procedure
- 5.1.5 Roots
- 5.1.6 Logarithms, powers and random digits
- 5.1.7 Powers with real exponents
- 5.1.8 Random number generators
- 5.1.9 Procedures from elementary number theory

5.2 The circle and the hyperbola

5.3 Roots of equations

5.4 Optimization (maxima and minima I-III)

5.5 Numerical integration

- 5.5.1 The initial value problem
- 5.5.2 Simulation of deterministic dynamic processes I-IV

5.6 Combinatorics

- 5.6.1 Procedures for the Pascal triangle
- 5.6.2 Frequency counts
- 5.6.3 Permutation problems

5.7 Probability problems

- 5.7.1 Birthday problem
- 5.7.2 Binomial distribution
- 5.7.3 Asymptotic formulas

5.8 Sorting

5.9 Simulation of random processes

- 5.9.1 Simulation with a random number generator
- 5.9.2 Symmetric random walk and the $\sqrt{n}$- law
- 5.9.3 Data analysis
- 5.9.4 Simulation without a random number generator

5.1 Program reading and program writing for lower grades

5.1.1 *Some simple, basic procedures*

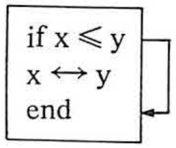
A small number of simple procedures occur very often. They are considered basic and get short identifying names. They are the building blocks for more sophisticated procedures.

1.

```
t ← x
x ← y
y ← t
```

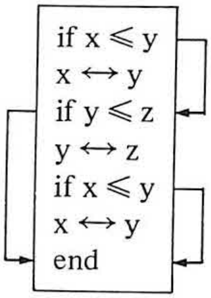
$\iff$ swap (x,y) $\iff$ $x \leftrightarrow y$

2. You have a two-pan scale and objects with weights x, y, z, u, ... You want to bring them into increasing order by pairwise comparisons. This is called *sorting*.



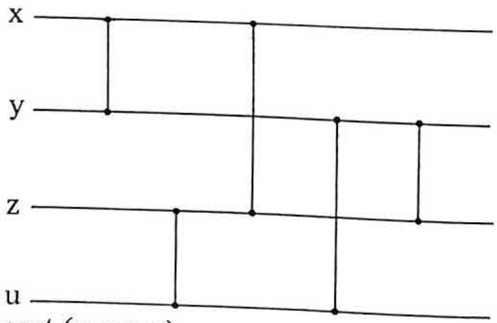
```
if x ≤ y
x ↔ y
end
```

sort (x,y)



```
if x ≤ y
x ↔ y
if y ≤ z
y ↔ z
if x ≤ y
x ↔ y
end
```

sort (x,y,z)

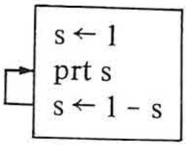


sort (x,y,z,u)

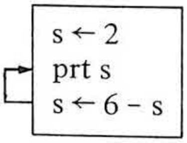
Deeper and systematic investigation of sorting procedures is postponed until much later.

3. Procedures for generating important sequences. Periodicity

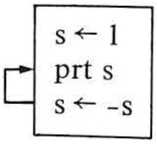
a) Switches, oscillators



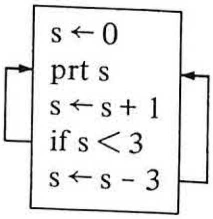
```
s ← 1
prt s
s ← 1 - s
```



```
s ← 2
prt s
s ← 6 - s
```



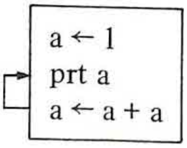
```
s ← 1
prt s
s ← -s
```



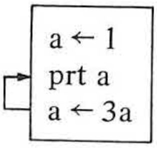
```
s ← 0
prt s
s ← s + 1
if s < 3
s ← s - 3
```

1,0,1,0,1,0, . . .
2,4,2,4,2,4, . . .
1,-1,1,-1,1, . . .
0,1,2,0,1,2, . . .

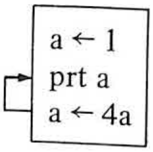
b) Powers



```
a ← 1
prt a
a ← a + a
```



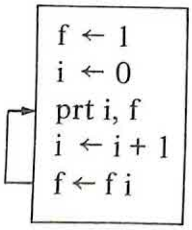
```
a ← 1
prt a
a ← 3a
```



```
a ← 1
prt a
a ← 4a
```

1,2,4,8,16,32, . . .
1,3,9,27,81,125, . . .
1,4,16,64,256,1024, . . .

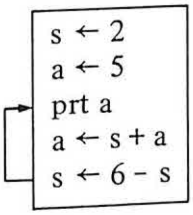
c) Table of factorials



```
f ← 1
i ← 0
prt i, f
i ← i + 1
f ← f i
```

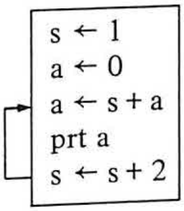
i	0	1	2	3	4	5	6	7	8	9	10
f	1	1	2	6	24	120	720	5040	40320	362880	3628800

d) Some additional remarkable sequences



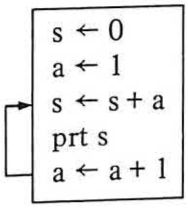
```
s ← 2
a ← 5
prt a
a ← s + a
s ← 6 - s
```

6n ± 1



```
s ← 1
a ← 0
a ← s + a
prt a
s ← s + 2
```

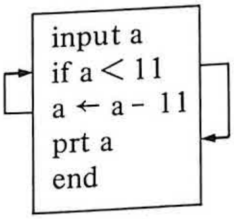
squares



```
s ← 0
a ← 1
s ← s + a
prt s
a ← a + 1
```

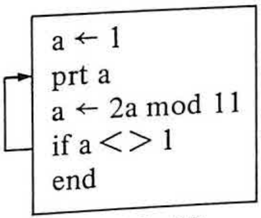
triangular numbers

5,7,11,13,17,19, . . . ,6n ± 1, . . .
1,4,9,16,25, . . . ,n^2, . . .
1,3,6,10,15, . . . ,n(n + 1)/2, . . .



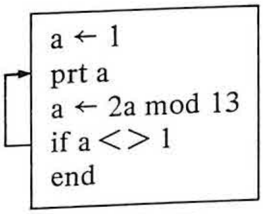
```
input a
if a < 11
a ← a - 11
prt a
end
```

mod 11



```
a ← 1
prt a
a ← 2a mod 11
if a <> 1
end
```

scramble 10



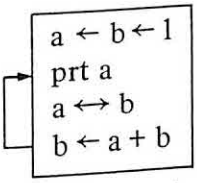
```
a ← 1
prt a
a ← 2a mod 13
if a <> 1
end
```

scramble 12

1,2,4,8,5,10,9,7,3,6

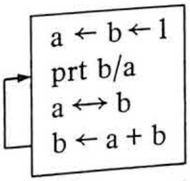
1,2,4,8,3,6,12,11,9,5,10,7

The scramblers are the first steps toward random permutations and random number generators. Design scramblers of your own! Replace line 3 of the scramblers by a ←3a, a ←4a, ... Which replacements give complete scrambles?



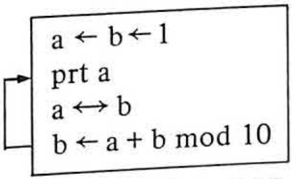
```
a ← b ← 1
prt a
a ↔ b
b ← a + b
```

Fibonacci



```
a ← b ← 1
prt b/a
a ↔ b
b ← a + b
```

golden ratio



```
a ← b ← 1
prt a
a ↔ b
b ← a + b mod 10
```

Fibonacci mod 10

The Fibonacci sequence modulo 10 is for empirical exploration. How does the period p depend on the modulus m = 2, 3, 4, . . . , 10? Conjectures!

Beware of exclusive use of flowcharts! It is not good for your mental health! Here is a procedure in plain English. Execute it. Translate it into a flowchart.

Example 1.

```
Start with a natural number n
while n <> 1 do
if n even then n ← n/2 else n ← 3n + 1
```

Example 2.

```
Start with a natural number n divisible by 3
repeat n ← sum of the cubes of the digits of n until n = 153
```

Example 3.

```
Start with a pair (a, b) of positive integers
while a <> 0 do
if a ⩾ b then (a, b) ← (a-b,2b) else (a, b) ← (2a, b-a)
```

Try to find all starting values for which the algorithm stops. What about the number of steps?

Example 4.

```
Start with integers a, b, c from 0 to 9. Set $(x, y, z) \leftarrow (a, b, c)$
repeat $(x, y, z) \leftarrow (y + z \bmod 9, x + z \bmod 9, x + y \bmod 9)$ until $(x, y, z) = (a, b, c)$
```

Try to find a simple rule which predicts the number of steps.

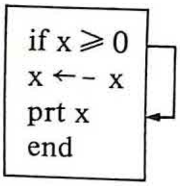
5.1.2 *Definition of functions by statements and flowcharts*

$\max(x, y) = \text{if } x \geqslant y \text{ then } x \text{ else } y$
$\text{abs}(x) = \max(x, -x) = \text{if } x \geqslant 0 \text{ then } x \text{ else } -x$
$\text{int}(x) = [x] =$ the largest integer $\leqslant x$. This is the most important function.

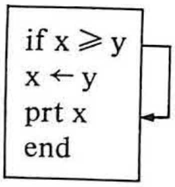
Every integer a can be uniquely expressed by the integer $b > 0$ in the form

$$a = bq + r, \quad 0 \leqslant r < b$$

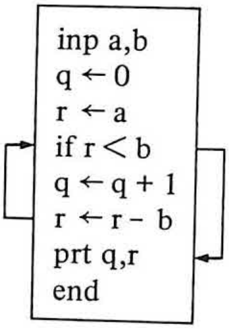
We have $q = \text{int}(a/b)$ and $r = a - b \times \text{int}(a/b) = a \bmod b$



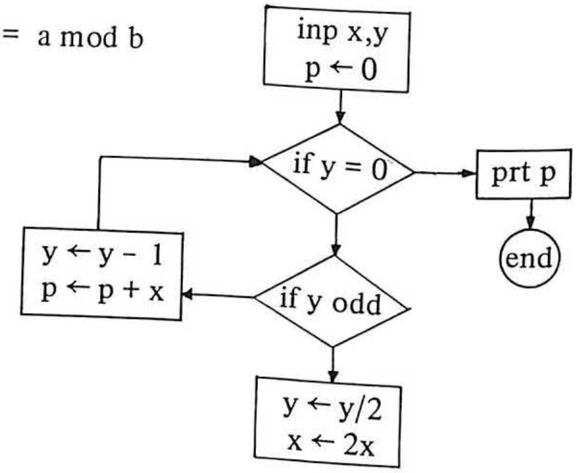
abs (x)



max (x,y)



(a,b) ↦ (q,r)



(x,y) ↦ x × y

5.1.3 *Ancient Egyptian multiplication procedure*

Scribes in ancient Egypt were skillful in addition, doubling and halving. How do you find the product p of x and y by using these simple operations only?

They used the fact that $xy = x(y-1) + x = (2x)(y/2)$ to reduce y as shown by the numerical example

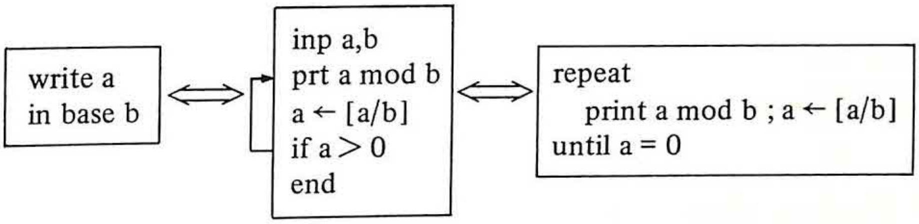
$$25 \times 19 = 25 \times 18 + 25 = 50 \times 9 + 25 = 50 \times 8 + 75 = 100 \times 4 + 75 = 200 \times 2 + 75 = 400 \times 1 + 75 = 475.$$

Simplify this procedure further as shown in the table. Simplify the flowchart accordingly.

$x \leftarrow 2x$	$y \leftarrow [y/2]$
25	19
50	9
~~100~~	4
~~200~~	2
400	1
475	

This program is important since a tiny modification gives the famous fast powering procedure.

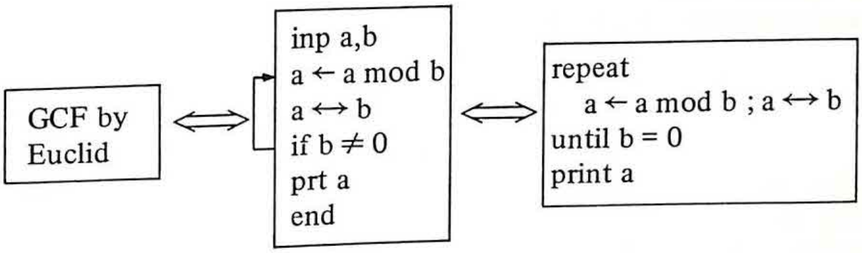
The next two basic procedures are quite famous:



```
write a
in base b
```

```
inp a,b
prt a mod b
a ← [a/b]
if a > 0
end
```

```
repeat
    print a mod b ; a ← [a/b]
until a = 0
```



```
GCF by
Euclid
```

```
inp a,b
a ← a mod b
a ↔ b
if b ≠ 0
prt a
end
```

```
repeat
    a ← a mod b ; a ↔ b
until b = 0
print a
```

5.1.4. *Growth, decay and the fast powering procedure*

a) You put an amount x into a bank. At the end of each year the bank makes the replacement

$$x \leftarrow qx\ , \quad \text{with } q = 1.05 \text{ being a typical value.}$$

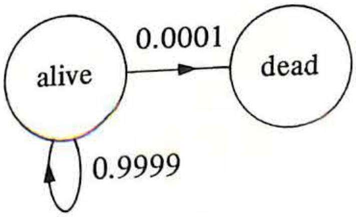
How long will it take until x is doubled, tripled, quadrupled, . . . ?

b) Right now the human race grows every year by 2%:

$$H \leftarrow 1.02\,H\,.$$

Suppose you start with Adam and Eve. How many years would it take to surpass the current world population of 4×10^9 ? You may assume constant growth rate 2% per annum (1082 years).

c) The chances to survive the crossing of a busy street are approximately 0.9999. The process of crossing can be represented by this figure.



What are the chances of surviving 10000 crossings? You must find 0.9999^{10000}. How do you find high powers a^n without tears? If the exponent n is a power of 2 then you simply square repeatedly. If $n = 2^k$ then you must press k times the square-key on the pocket calculator. But what if n is not a power of 2? For example, how do you find a^{10}? We can find a^{10} by the following procedure:

$$p \leftarrow aa\ ;\ p \leftarrow pp\ ;\ p \leftarrow pa\ ;\ p \leftarrow pp\,.$$

By applying this procedure four times we get a^{10000}.

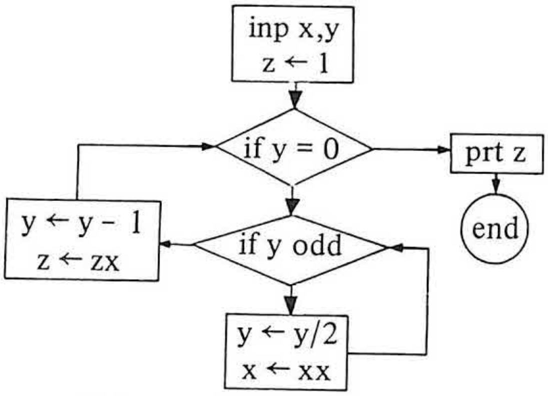
Special purpose procedures such as these are quite instructive, but of limited applicability. We are looking for a fast universal powering program. The following numerical example shows such a procedure:

$$1 \times 2^{15} = 2 \times 2^{14} = 2 \times 4^7 = 8 \times 4^6 = 8 \times 16^3 = 128 \times 256 = 32\,768 .$$

It uses two transformations, which leave the product $z \times x^y$ invariant:

$$\text{if } y \text{ is odd do } y \leftarrow y-1;\ z \leftarrow zx \text{ else do } y \leftarrow y/2;\ x \leftarrow x^2 .$$

To execute this algorithm on a pocket calculator you need at least two memories, one for x and one for z. We can keep track of y on a slip of paper. Later this procedure will be modified slightly to handle real exponents. But before doing this we must deal with square roots.



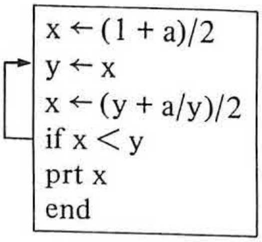
fast powering procedure

5.1.5 *Roots*

a) The square root operation is fundamental. The computation of all other roots and of many elementary functions can be reduced to repeated square root extractions.

We construct a procedure with input $a > 0$ and output $\sqrt{a}$. The procedure generates a decreasing sequence x_n with superfast convergence to $\sqrt{a}$. The sequence is defined by the recurrence

$$x_o = (1 + a)/2 , \quad x_{n+1} = (x_n + a/x_n)/2 , \quad n \geqslant 0 .$$



$a \mapsto \sqrt{a}$

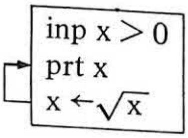


Fig. 1

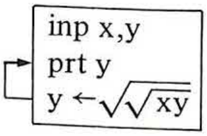


Fig. 2

b) The tiny procedure in Fig. 1 prints a sequence which converges to 1. Experiment with your pocket calculator. You will find that near $x = 1$ the convergence is linear with rate 1/2, i.e. $(1 - x_{n+1})/(1 - x_n) \to 1/2$. This is easy to prove. If your aim is to compute the limit 1 you get one bit per iteration and you will need 32-33 iterations for a 10-digit accuracy.

If there is a $\sqrt{\ }$ - key on your pocket calculator then you can easily find $\sqrt[n]{x}$ for any n. Let us design a procedure for $y = \sqrt[3]{x}$. For $y > 0$ we have

$$y = \sqrt[3]{x} \iff y^3 = x \iff y^4 = xy \iff y = \sqrt{\sqrt{xy}}.$$

The equation $y = \sqrt{\sqrt{xy}}$ will be solved by iteration. Fig.2 has input x and an estimate $y_o > 0$ of y, which may be arbitrary. The procedure prints the sequence

$$y_o, y_1 = \sqrt{\sqrt{xy_o}}, \; y_2 = \sqrt{\sqrt{xy_1}}, \; \dots .$$

It is easy to show that $y_n \to \sqrt[3]{x}$ with linear rate 1/4. Thus about 16 iterations are needed for a 10-digit accuracy. Similarly one can deal with other roots. For example:

$$y = \sqrt[5]{x} \iff y^5 = x \iff y^{15} = x^3 \iff y^{16} = x^3 y \iff y = \sqrt{\sqrt{\sqrt{x\sqrt{xy}}}}$$

$$y = \sqrt[7]{x} \iff y^7 = x \iff y^8 = xy \iff y = \sqrt{\sqrt{\sqrt{xy}}}$$

$$y = \sqrt[9]{x} \iff y^9 = x \iff y^8 = x/y \iff y = \sqrt{\sqrt{\sqrt{x/y}}}$$

The iterations are easy to execute on the pocket calculator. The convergence rates 1/16, 1/8, 1/8 can be *observed* on the display. This observation makes the concept of convergence rate intuitively clear. Later algorithms for elementary functions will be rated according to their convergence speed.

5.1.6 *Logarithms, powers and random digits*

For $1 < a < b$ we want to solve the equation

$$a = b^x ,$$

for $x = \log_b a$, $0 < x < 1$. We write x in base 2:

$$x = 0.x_1 x_2 x_3 \dots = x_1/2 + x_2/4 + x_3/8 + \dots .$$

We can get the digits $x_1, x_2, x_3, \dots$ one by one by repeatedly squaring, as shown in Fig. 3. This procedure yields one bit per iteration. Yet it is extremely fast, and there is an unexpected bonus. It generates binary random digits. Thus with a pocket calculator we can get one toss of a fair coin per second. For instance, $a = 1.234567891$, $b = 2$ yields

```
                                                     10000   11000   10100
                                     10101   10010   10100   00000   01011
x  = 0.01001   10111   01001        11001   11100   10101   10111   11001
       10100   00110   10010        01101   11100   00001   11100   10100
       11101   11101   11111        10110   01110   00010   10001   00100 . . .
       00001   11111   11000        01100   11100
       01100   11000   00101
```

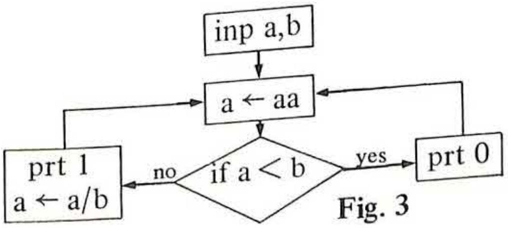


Fig. 3

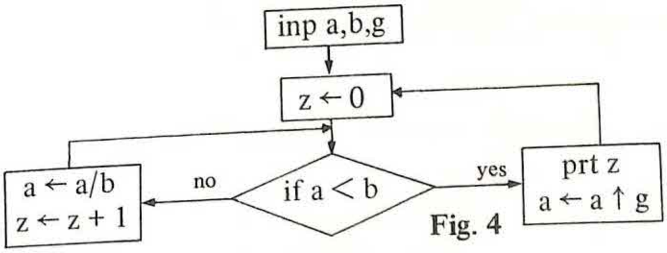


Fig. 4

Similarly, Fig. 4 is a generator of random digits from $\{0, 1, 2, \dots, g - 1\}$, but it is only competitive for small g and fast powering. This procedure computes $\log_b a$ in base g.

5.1.7 *Powers with real exponents*

Given $a > 0$ and $0 < x < 1$. Find $y = a^x$.

We write x in base 2:

$$x = 0.x_1 x_2 x_3 \dots = x_1/2 + x_2/4 + x_3/8 + \dots .$$

Then $$y = a^x = a^{x_1/2} \times a^{x_2/4} \times a^{x_3/8} \dots .$$

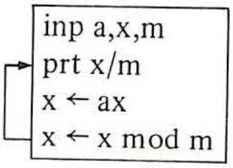
Thus we can find a^x by repeated extraction of square roots.

5.1.8 *Random number generators*

Discuss the random selection of a point from (0, 1), which is called a *random number*, and the selection of an element of $\{0, 1, 2, \dots, n-1\}$ which is called a *random n-digit*.

Weakness of the middle-square procedure.

The linear congruential procedure (Fig. 5). For input use odd x, $a \equiv -3$ modulo 8, and $m = 2^{17}$.



```
inp a,x,m
prt x/m
x ← ax
x ← x mod m
```

Fig. 5

5.1.9 *Procedures from elementary number theory*

Conversion of numbers from any base to any other base.

Euclidean algorithm and Pythagorean triples.

Extended Euclidean algorithm, i.e. find $d = \text{GCF}(a,b)$ and integers x, y such that $ax + by = d$.

Continued fractions and their relation to the Euclidean algorithm.

Periodic decimals. Write a program which for any $n < 100$, n prime to 10, prints the length of the period of $1/n$. Patterns and conjectures.

Prime numbers. Prime number table by division. Fast twin prime program. Factoring program. *Sieve of Eratosthenes*:

1. Set $p \leftarrow 2$
2. Strike all multiples of p after p.
3. Set $p \leftarrow$ the first surviving number after p and go to 2.

5.2 The circle and the hyperbola

The unit circle $x^2 + y^2 = 1$ and the unit hyperbola $xy = 1$ are of fundamental importance. With their help we can define all the elementary transcendental functions which occur in senior high school and in college calculus courses. The definitions are geometric and constructive. They lead to short and elegant algorithms for the computation of these functions.

a) Fig. 6 shows the unit circle; c and c_1 are the distances of the chords BD and CD from the center; 2s and $2s_1$ are the lengths of these chords. The relations

$$AC \times DE = AD \times CD \text{ and } AD^2 = AE \times AC$$

yield

$$s_1 = s/2c_1\ ,\ c_1 = \sqrt{(1 + c)/2}\ .$$

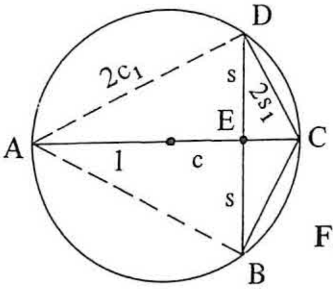


Fig. 6

These relations easily lead to an algorithm for the efficient computation of arcsin x, arccos x, arctan x, arccot x with the pocket calculator, as shown in Fig. 7 and Fig. 8.

```
inp s,c
c ← $\sqrt{(1 + c)/2}$
s ← s/c
if c < 1
prt s
end
```

Fig. 7

input s	input c	output	domain
x	$\sqrt{1 - x^2}$	arcsin x	$-1 \leqslant x \leqslant 1$
$\sqrt{1 - x^2}$	x	arccos x	$-1 < x \leqslant 1$
$\dfrac{x}{\sqrt{1 + x^2}}$	$\dfrac{1}{\sqrt{1 + x^2}}$	arctan x	$-\infty < x < \infty$
$\dfrac{1}{\sqrt{1 + x^2}}$	$\dfrac{x}{\sqrt{1 + x^2}}$	arccot x	$-\infty < x < \infty$

Fig. 8

```
inp s,c
$s_1$ ← s
c ← $\sqrt{(1 + c)/2}$
s ← s/c
if 1 - c > 1E - 5
prt (4s - $s_1$)/3
end
```

Fig. 9

The algorithm has linear convergence rate 1/4, so we need about 16 iterations to get 10-digit accuracy. With no additional effort we can get convergence rate 1/16, so that 8 iterations will suffice for 10-digit accuracy. Fig. 9 shows one possibility based on Huygens' extrapolation. By the way, the input $s = 2$, $c = 0$ yields π.

Now we design a fast sine-procedure. It is based on a little known theorem of elementary geometry with a delightfully simple proof. In Fig. 10, $S = s(3 - s^2)$.

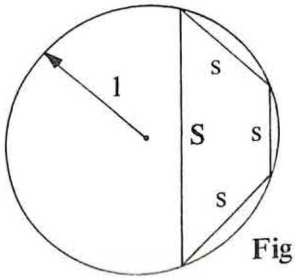


Fig. 10

```
inp x,n
s ← 2x/3↑n
s ← s (3 - ss)
n ← n - 1
if n > 0
prt s/2
end
```

Fig. 11

(Of course this is $\sin 3x = 3\sin x - 4\sin^3 x$ in disguise.) From the length $2x$ of an arc we must find the length $2\sin x$ of its chord. We subdivide the arc $2x$ into 3^n equal parts. If n is "sufficiently large" (which can be made precise) the arc is practically equal to its chord. Thus initially we set $s \leftarrow 2x/3^n$ and apply $s \leftarrow s\,(3 - ss)$ n times. Then $s = 2\sin x$ approximately. The sine-procedure in Fig. 11 is based on this idea. By numerical experimentation with the test cases $x = \pi/6$, $x = \pi/3$, $x = \pi/2$ we find that $n = 9$ guarantees an error $\leqslant 10^{-10}$. By changing the print statement appropriately we get cos x, tan x, cot x. Again this procedure is easily executed by means of a pocket calculator.

b) Fig. 12 shows the hyperbola $xy = 1$. Define ln x as the shaded area and $\exp = \ln^{-1}$. By proceeding in complete analogy with the circle one gets the procedure in Fig. 13 which computes ln x and the four inverse hyperbolic functions as shown by Fig. 14.

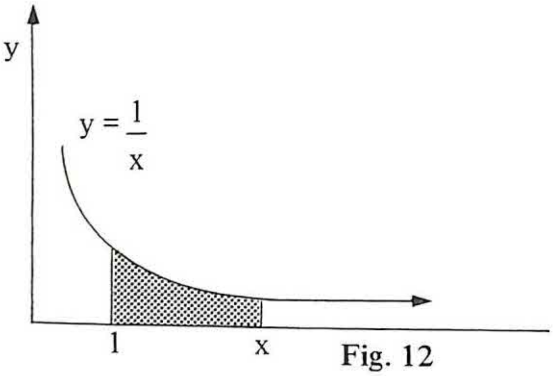


Fig. 12

```
inp s,c
  c ← $\sqrt{(1 + c)/2}$
  s ← s/c
  if c > 1
prt s
end
```

Fig. 13

input			
s	c	output	domain
$\frac{1}{2}(x - \frac{1}{x})$	$\frac{1}{2}(x + \frac{1}{x})$	ln x	$x > 0$
x	$\sqrt{1 + x^2}$	arsinh x	$-\infty < x < \infty$
$\sqrt{x^2 - 1}$	x	arcosh x	$x \geqslant 1$
$\frac{x}{\sqrt{1 - x^2}}$	$\frac{1}{\sqrt{1 - x^2}}$	artanh x	$-1 < x < 1$
$\frac{1}{\sqrt{1 - x^2}}$	$\frac{x}{\sqrt{1 - x^2}}$	arcoth x	$-1 < x < 1$

Fig. 14

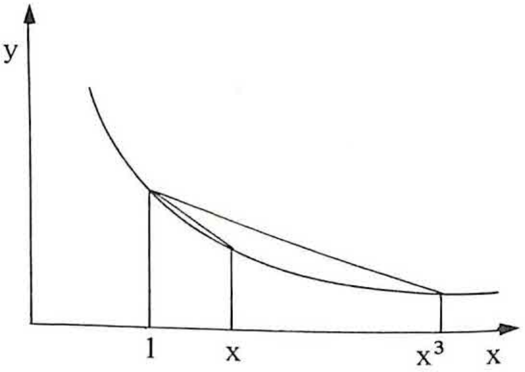


Fig. 15

In Fig. 15 denote the areas of the trapezoids from 1 to x and 1 to x^3 by s and S, respectively. Then $S = s\,(3 + 4s^2)$. This identity can be used to construct the procedure in Fig. 16 for exp x. By printing s, $\sqrt{1 + s^2}$, $s/\sqrt{1 + s^2}$, $\sqrt{1 + s^2}/s$ we get sinh x, cosh x, tanh x, coth x. Again $n = 9$ is sufficient for 10-digit accuracy.

```
inp x,n
s ← x/3↑n
s ← s(3 + 4ss)
n ← n - 1
if n > 0
prt s + √(1 + ss)
end
```

Fig. 16

In secondary school one will usually restrict oneself to the functions ln and exp.

The two algorithms in Fig. 7 and Fig. 13 can be united if we replace line 4 by

$$\text{if } |1 - c| > \epsilon$$

for some permissible error ϵ, for instance $\epsilon = 10^{-10}$. Then the same procedure computes 9 functions.

5.3 Roots of equations

Bisection method, regula falsi, iteration $x \leftarrow g(x)$ for contracting mappings g, Newton's method, inverse of a function by Newton's method.

5.4. Optimization (maxima and minima I-III)

I. Maxima and minima of an array

Find a) maxima, b) minima, c) maxima and minima of an array.

Solution to a) (leftmost absolute maximum):

```
i ← 1
for k ← 2 to n do
    if a(k) > a(i) then i ← k.
```

II. Maximum of a unimodal function f in [a, b].

a) Trisection method

```
1.  h ← b - a
2.  if h < ε then output (a + b)/2
3.  u ← a + h/3; v ← b - h/3
4.  if f(u) < f(v) then a ← u else b ← v
5.  go to 1.
```

b) Bisection method

```
1.  c ← f(m)
2.  if f(m - h/2) > c then m ← m - h/2
3.  if f(m + h/2) > c then m ← m + h/2
4.  h ← h/2
5.  if h > ε then 1 else output m, f(m).
```

c) Golden Search

```
1.   While $b - a > \epsilon$ do
     1.1. $a_1 \leftarrow a + \frac{3-\sqrt{5}}{2} \cdot (b - a)$;  $b_1 \leftarrow a + \frac{\sqrt{5}-1}{2} \cdot (b - a)$
     1.2. if $f(a_1) \leqslant f(b_1)$ then $a \leftarrow a_1$ else $b \leftarrow b_1$
2.   prt $(a + b)/2$.
```

III. Search in the Plane

An unmanned vehicle lands on an unknown planet. It can stretch out feelers in any direction and can measure the slope in that direction. Write procedures steering it onto a peak to plant a flag and into a deep spot in search of water. Discuss several standard methods and give examples of treacherous terrain for which the methods fail. The shape of the terrain is stored as a function $z = f(x, y)$.

a) Random search,
b) Gauss-Seidel (search parallel to the axes),
c) Cauchy's steepest descent, etc.

5.5 Numerical integration

No calculus is needed for this topic! Trapezoidal rule, midpoint rule, Simpson's rule, Romberg scheme – these are enough. Simpson's rule and the Romberg scheme can be developed by numerical experimentation. This is very interesting and instructive. For the Romberg scheme the pocket calculator should have at least four, preferably even more registers.

5.5.1 *The initial value problem*

If the students know a tiny bit of calculus, then solve the *initial value problem*

$$(1) \quad y' = f(x, y), \quad y(x_o) = y_o,$$

by Euler's method, the trapezoid method and midpoint method. This can be done even if the students are not familiar with calculus. Explain intuitively the idea of a direction field. Now (1) describes a motion in the plane with continuously varying direction. Simulate this motion by a discrete process.

```
$x \leftarrow x_0$ ; $y \leftarrow y_0$
→ prt x,y
  $y \leftarrow y + hf(x,y)$
  $x \leftarrow x + h$
```

Euler

```
$x \leftarrow x_0$ ; $y \leftarrow y_0$
→ prt x,y
  $a \leftarrow x + h$
  $b \leftarrow y + hf(x,y)$
  $y \leftarrow y + \frac{h}{2}(f(x,y) + f(a,b))$
  $x \leftarrow a$
```

Trapezoid

```
$x \leftarrow x_0$ ; $y \leftarrow y_0$
→ prt x,y
  $a \leftarrow x + h/2$
  $b \leftarrow y + \frac{h}{2}f(x,y)$
  $x \leftarrow x + h$
  $y \leftarrow y + hf(a,b)$
```

Midpoint

5.5.2 *Simulation of deterministic dynamic processes I-IV*

No calculus is needed for these problems of numerical integration. The only prerequisites are similarity and the Pythagorean theorem. Some vector algebra is quite useful.

I. Pursuit problems. Two typical problems will be mentioned.

a) A rabbit runs due north and a dog chases the rabbit always heading toward the rabbit. Plot the dog's path.

b) The dog D chases the rabbit R always heading toward the rabbit. The rabbit evades the dog by always running at right angles to the line DR in Fig. 17. Plot the paths of dog and rabbit.

Solution for b): The current positions of dog and rabbit are $\vec{D} = (x,y)$ and $\vec{R} = (u,v)$, and one second later $\vec{D}_1 = (x_1,y_1)$, and $\vec{R}_1 = (u_1,v_1)$. Let $d = |DR|$ and let p and q be the velocities of dog and rabbit. Then we have the procedure in Fig. 18.

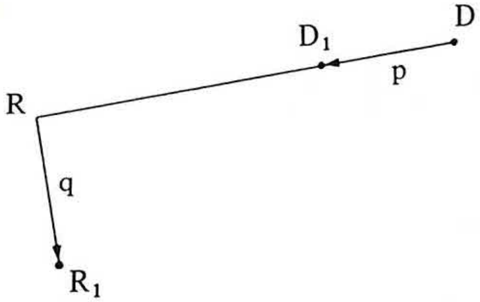


Fig. 17

```
inp $\vec{D},\vec{R},p,q$
prt $\vec{D},\vec{R},d$
$\vec{D}_1 \leftarrow \vec{D} + \frac{p}{d}(u - x, v - y)$
$\vec{R}_1 \leftarrow \vec{R} + \frac{q}{d}(-v + y, u - x)$
$\vec{D} \leftarrow \vec{D}_1$ ; $\vec{R} \leftarrow \vec{R}_1$
```

Fig. 18

II. Planetary motion (No calculus, please!)

At O there is a heavy body of mass M and at $\vec{R} = (x,y)$ there is a light satellite with mass m and velocity $\vec{v} = (p,q)$. At time t + h we have $\vec{R}_1 = (x_1,y_1)$, $\vec{v}_1 = (p_1,q_1)$. The satellite is attracted to O with force

$$\vec{F} = -\frac{GmM}{r^3}\vec{R}, \quad r = |\vec{R}|, \ G = \text{gravitational constant.}$$

The acceleration of the satellite is

$$\vec{a} = (\vec{v}_1 - \vec{v})/h .$$

Newton's second axiom $\vec{F} = m\vec{a}$ yields

$$\vec{v}_1 = \vec{v} - \alpha h\vec{R}/r^3 , \quad \alpha = GM.$$

Fig. 19 shows that $\vec{R}_1 = \vec{R} + h\vec{v}$. Thus we have the procedure in Fig. 20. There are many interesting modifications. For instance motion of a satellite under the influence of strong "solar wind" with amazing fascinating orbits.

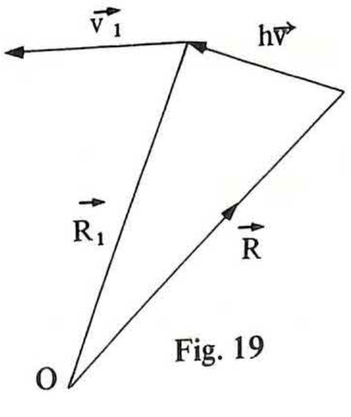


Fig. 19

```
inp $\vec{R},\vec{v},h$
prt $\vec{R},\vec{v}$
$\vec{R}_1 \leftarrow \vec{R} + h\vec{v}$
$\vec{v}_1 \leftarrow \vec{v} - \alpha h\vec{R}/r^3$
$\vec{R} \leftarrow \vec{R}_1$
$\vec{v} \leftarrow \vec{v}_1$
```

Fig. 20

III. Spread of rumours and epidemics

IV. Predator-prey-problems

5.6 Combinatorics

5.6.1 *Procedures for the Pascal Triangle*

PASCAL I is based on the formula $\binom{n}{s} = 1 \cdot \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \ldots \cdot \frac{n-s+1}{s}$

and uses no array.

PASCAL II is based on $\binom{n}{0} = \binom{n}{n} = 1$ and $\binom{n}{s} = \binom{n-1}{s-1} + \binom{n-1}{s}$

and uses a two-dimensional array.

PASCAL III is based on the property exemplified by

$$\begin{array}{ccccccc} 1 & 5 & 10 & 10 & 5 & 1 & \\ & 1 & 5 & 10 & 10 & 5 & 1 \\ \hline 1 & 6 & 15 & 20 & 15 & 6 & 1 \end{array}$$

and uses a one-dimensional array.

5.6.2 *Frequency counts (deeply nested loops)*

A die is rolled n times. How many of the 6^n possible outcomes have point sum n, n + 1, . . . , 6n?

Money change problems. In how many ways can you change a dollar bill? And related problems.

5.6.3 *Permutation problems*

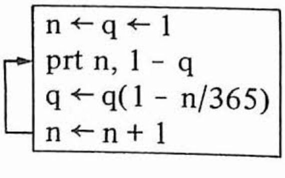
Generation of a random permutation, inverse of a permutation, order of a permutation, decomposition into cycles, Josephus-problem, etc.

5.7 Probability problems

Some elementary probability problems require extensive computations. These are ideal computer problems. Three typical examples:

5.7.1 *Birthday problem*

In a room there are n persons. What is the probability that at least two of the persons have the same birthday?



```
n ← q ← 1
prt n, 1 - q
q ← q(1 - n/365)
n ← n + 1
```

Fig. 21

The procedure in Fig. 21 gives the answer for each n. It is easy to execute with the pocket calculator.

5.7.2 *Binomial distribution*

Let p and q = 1 - p be the probabilities of success and failure in a Bernoulli process. Then the probability of x successes in n trials is

$$b(x) = \binom{n}{x} p^x q^{n-x}, \quad x = 0, 1, 2, \ldots, n.$$

A typical probability problem is to find the probability of a or less successes. This leads to the sum $s = b(0) + b(1) + \ldots + b(a)$, which cannot be computed by hand. The procedure in Fig. 22 computes s without tears. It is based on the recursion

$$b(0) = q^n, \quad b(x) = \frac{n-x+1}{x} \cdot \frac{p}{q} \cdot b(x-1), \quad x = 1, 2, \ldots, n.$$

```
10  INPUT A,N,P
20  Q = 1 - P
30  S = B = Q↑N
40  FOR X = 1 TO A
50        B = B*P*(N - X + 1)/(Q*X)
60        S = S + B
70  NEXT X
80  PRINT A,N,P,S
90  END
```

Fig. 22

5.7.3 *Asymptotic formulas*

If p = q = 1/2 then the probability of n successes in 2n trials is

$$b(n) = \frac{1 \times 3 \times 5 \times \ldots \times (2n-1)}{2 \times 4 \times 6 \times \ldots \times 2n}.$$

By numerical experimentation with a computer it is possible to discover the remarkable asymptotic formula

$$b(n) \sim \frac{1}{\sqrt{\pi n}} \left(1 - \frac{1}{8n} + \frac{1}{128n^2}\right).$$

5.8 Sorting

Sorting by insertion, sorting by selection, sorting by interchange, sorting by frequency counts. These are $0(n^2)$ methods and thus unsuitable for large n. Discuss one example of an $0(n \log n)$-algorithm. For example, explain merge-sort without necessarily programming it. Show that for $n = 2^m$ only $n\log_2 n - n + 1$ comparisons are needed for sorting n items.

Discuss several big programs, like programming big games, the eight queens problem.

5.9 Simulation of random processes

Probability is an inexhaustible source of top quality computer problems. Simulation programs are especially instructive. In addition *they do not require any prerequisites*. We do not envisage the use of a powerful simulation language like SIMULA or SMALLTALK. One learns more if one accomplishes a task with limited means, like BASIC. There are two types of simulations: those requiring the random number generator and those that do not. See Engel (1975).

5.9.1 *Simulation with a random number generator*

At the instruction *rnd* the computer generates a real number u which is uniformly distributed in the interval (0, 1). This is the raw material for imitating random processes. By suitable transformations of u we can simulate any random device.

Tosses of a fair coin: [2rnd] or [rnd + 0.5] produce 0 or 1 with probability 1/2.

Steps of a symmetric random walk: 2[2rnd] − 1 or 2[rnd + 0.5] − 1 generate 1 or − 1 with probability 1/2.

Unfair coin: [rnd + p] assumes the values 1 or 0 with probability p and q = 1−p, respectively.

[n × rnd] selects at random one element from $\{0, 1, 2, \ldots, n-1\}$.

Draw a random s-sample from an n-set. The elements of the set are stored in L(1), L(2), . . . , L(n).

Draw at random elements from $\{1, 2, \ldots, n\}$ until you have a *complete set* of all elements. Estimate the expected number of draws.

Write a program for the game of Craps. The computer should play the game 1000 times and print estimates for the winning probability and the expected duration of a game.

Study runs in random binary sequences.

Generate 1000 random decimal digits and test their quality by χ^2.

Relative frequency and probability: Write a program which tosses a fair coin 2000 times, print the relative frequency s/n of successes for n = 10, 20, 30, . . . , 100, 200, 300, . . . , 2000 and the deviation $|s/n - 0.5|$ in σ-units ($\sigma = 1/2\sqrt{n}$).

5.9.2 *Symmetric random walk and the $\sqrt{n}$ - law*

Simulate random walks on the line, in the plane and in space, starting at the origin and discover empirically that after n steps the square of the expected distance from the origin is equal to n.

```
x ← 0
  x ← x + 2[2rnd] - 1
```

Random walk on a line

```
x(1) ← x(2) ← 0
  a ← [2rnd] + 1
  x(a) ← x(a) + 2[2rnd] - 1
```

Random walk in the plane

```
x(1) ← x(2) ← x(3) ← 0
  a ← [3rnd] + 1
  x(a) ← x(a) + 2[2rnd] - 1
```

Random walk in space

```
x ← y ← 0
  a ← 2πrnd
  x ← x + cos a
  y ← y + sin a
```

Random walk in the plane with unit steps in random direction

Diffusion and Brownian motion.

5.9.3 *Data analysis*

The random numbers as a source of data with lucid structure. Generate a sequence u_1, u_2,

$u_3, \ldots$ of random numbers. Study records in the sequence. Introduce stopping rules:

Stop as soon as $u_{n-1} < u_n$

Stop as soon as $u_1 + u_2 + u_3 + \ldots + u_n > 1$

Stop as soon as $u_n > u_1$

What is the distribution and the expected value of the stopping time? The results are highly instructive and amazing.

How do you choose a random direction in 3-space?

Simulate radioactive decay. This does require prerequisites, especially the functions exp and ln.

A radioactive substance decays by sending out signals with gaps $T_1, T_2, T_3, \ldots$ between signals. The following program simulates a sequence of gaps:

```
T ← - ln(rnd)/λ
prt T
```

Proof: $$T < t \iff -\frac{1}{\lambda}\ln(\text{rnd}) < t \iff \ln(\text{rnd}) > -\lambda t \iff \text{rnd} > e^{-\lambda t}$$

$$P(T < t) = P(\text{rnd} > e^{-\lambda t}) = 1 - e^{-\lambda t}$$

5.9.4 *Simulation without a random number generator*

In school we treat almost exclusively discrete random processes (Markov chains), which can be represented by transition graphs. Such a process can be regarded as a discrete (time invariant) linear system. The linear system can be operated by the computer, or pocket calculator if enough memories are available. The outputs of the linear system are probabilities, distributions, expectations. We give only trivial illustrations. Details can be found in (Engel, 1976, 1977).

(a) Take the simplest Markov chain: tossing a fair coin until the first success comes up. It can be represented by Fig. 23.

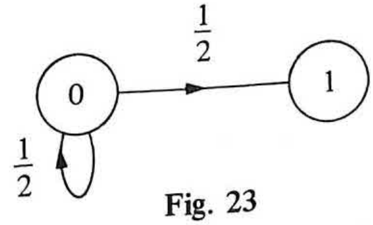


Fig. 23

```
q ← 1
c ← 0
prt q,c
c ← c + q
q ← q/2
```

Fig. 24

```
a ← 1
b ← c ← 0
c ← c + a + b
(a,b) ← $\frac{1}{2}$(a + b,a)
if a + b > 1E - 9
prt c
end
```

Fig. 26

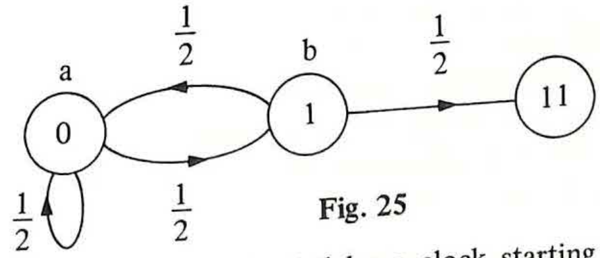


Fig. 25

This linear system is controlled by a clock starting at time $t = 0$. Initially mass $q = 1$ is placed in state 0. At time 1, 2, 3, . . . the mass is redistributed as shown by the labels of the branches. It costs 1 monetary unit to move mass 1 for 1 time unit. Let c cumulate the cost of operating the system. Then we have the procedure in Fig. 24. $c = 1 + 1/2 + 1/4 + \ldots$ converges to the expected waiting time for a success. After 32 iterations we get $c = 2$ accurate to 10 digits.

(b) Take the process of tossing a coin until a run of two successes comes up. It can be represented by the linear system in Fig. 25 and by the procedure in Fig. 26. This procedure prints the expected waiting time 6. Convergence is linear with rate $(\sqrt{5} - 1)/2$, i.e. rather slow.

This approach avoids matrices which cannnot be handled by means of the pocket calculator. Even more convenient is the use of the difference equation of the linear system. Take the Markov chain in Fig. 27, which is an example from genetics (see Engel, 1976). Suppose we start in state 1. Let p_n be the probability of absorption in exactly n transitions. Then

$$p_1 = 0\,,\ p_2 = 1/8\,,\ p_n = p_{n-1} - p_{n-2}/8\,,\ n > 2.$$

This difference equation can easily be treated by means of a pocket calculator.

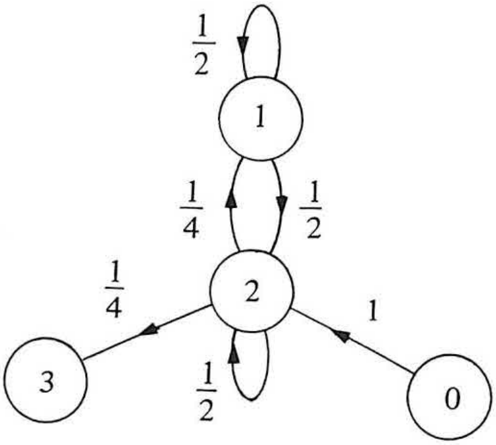


Fig. 27

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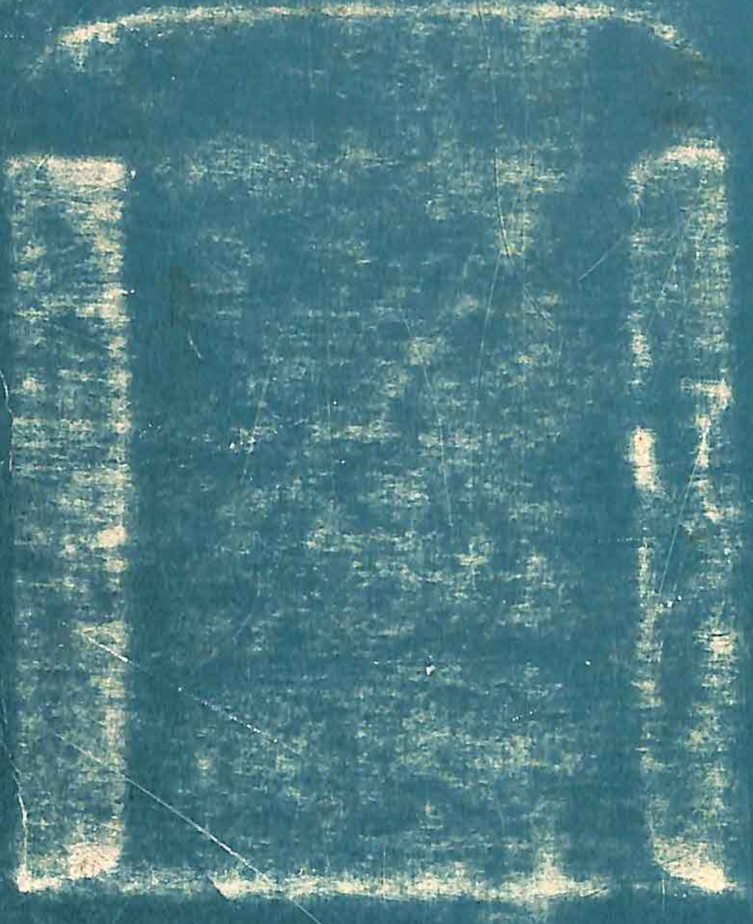

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